Physical Determination of the Action

Bruno Hartmann
Perimeter Institute for Theoretical Physics,
Waterloo, ON N2L 2Y5, Canada,
Humboldt University, D-12489 Berlin, Germany

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Abstract: Physical theories ought to be built up from colloquial notions such as 'long bodies', 'energetic sources' etc. in terms of which one can define pre-theoretic ordering relations such as 'longer than', 'more energetic than'. One of the questions addressed in previous work is how to make the transition from these pre-theoretic notions to quantification, such as making the transition from the ordering relation of 'longer than' (if one body covers the other) to the notion of *how much* longer. In similar way we introduce dynamical notions 'more impulse' (if in a collision one object overruns the other) and 'more energetic' (if the effect of one source exceeds the effect of the other). In a physical model - built by coupling congruent standard actions - those basic pre-theoretic notions become measurable. We uncover the origin of (basic) physical quantities of Energy, Momentum and Inertial Mass. From physical and methodical principles - without mathematical presuppositions - we derive all equations of (classical and relativistic) Dynamics and ultimately the principle of least action.

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1 Prelude

'Dynamics is the science of (movement) actions of forces' - Galilei defines [27] and - 'gets to the bottom of the free play of natural forces.' Newton 1687 provided an axiomatic system for Classical Mechanics [1]. He introduced the notions force and inertial mass by *definition* and the basic law of mechanics as a *postulate*. Beginning 1734 with Euler [2] attempts have been undertaken to *justify* those abstract notions 'inertial mass' and 'external force' with regard to operations with physical objects. Their fundamental equation though remained in the status of a postulate and that did not change ever since. The quest for a consistent set of basic dynamical measures for Classical and for Relativistic Mechanics [33] remains contested and leads - when based on the initial postulate of fundamental equations (Newton: $F = m \cdot a$, Euler: $F \cdot \Delta t = m \cdot \Delta v$ etc.) - into circular arguments.

There is no quantification of 'force' as a basic physical measure.[1] An alternative position regards notions mass and force inherit to the totality of an axiomatic system. According to this view one might determine force from known inertial mass by means of measuring acceleration provided one postulates a basic dynamical law

$$\begin{array}{lccc} \text{Newton II}: & F & := & m \cdot a \\ \text{Lorentz}: & F^{\nu}_{\mu} v^{\mu} & := & m \cdot a^{\nu} \end{array} \quad .$$

Despite the ambiguity (if velocity dependence is attributed to mass or force) the quantification rests on reliable basic measurements for inertial mass. Dating back to Euler [2] and popularized by Mach [7] inertial mass could be specified from colliding two objects $\mathcal{A}$ and $\mathcal{B}$

$$m_{\mathcal{A}} \cdot \Delta v_{\mathcal{A}} \stackrel{\text{(Newton II)}}{=} \left(F_{\mathcal{A}} \cdot \Delta t \stackrel{\text{(Newton III)}}{=} -F_{\mathcal{B}} \cdot \Delta t \right) \stackrel{\text{(Newton II)}}{=} -m_{\mathcal{B}} \cdot \Delta v_{\mathcal{B}}$$

by eliminating the undetermined force provided one further postulate Newton III. According to this hypothetical definition the ratio of the two inertial masses would be determined from the actual behavior in an interaction by the inverse ratio of their velocity changes

$$\frac{m_{\mathcal{A}}^{\text{(inert)}}}{m_{\mathcal{B}}^{\text{(inert)}}} := -\frac{\Delta v_{\mathcal{B}}}{\Delta v_{\mathcal{A}}} \quad . \tag{1}$$

Consider a simple application where we have a reservoir with equivalent physical objects $\{①\}$. Suppose we can tightly connect two of them $①*①$ such that the composite acts like a single rigid body (see figure 1). From the collision experiment between ① and $①*①$ we

[1] We distinguish basic and derived measures. In a *basic* measurement (i.e. of length or duration) the result does not require prior knowledge of the quantification of any other observables (provided a rigid meter-stick one simply counts the number of steps or provided a functioning clock [22] one counts the number of ticks). Instead e.g. a measurement of force according to Hooke's empirical law by means of deforming a spring requires knowledge (by means of what?) of the non-linear (!) expansion coefficient of the spring. Likewise for any proposal of a basic measurement for force - please check that the implicit assumptions are not circular.

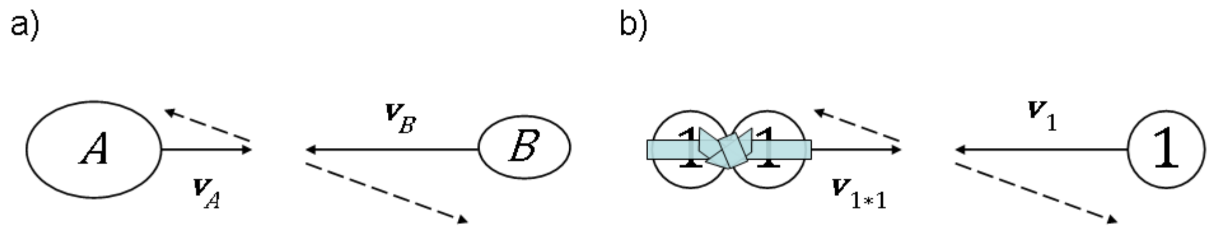

Figure 1: a) collision between two generic objects $\mathcal{A}$ and $\mathcal{B}$ b) in a reservoir of equivalent objects $\{\textcircled{1}\}$ two are bound together (e.g. by a practically massless sling) and collide as a composite $\textcircled{1}*\textcircled{1}$ with individual object $\textcircled{1}$

would quantify the ratio of their inertial masses

$$m^{(\text{inert})}_{\textcircled{1}*\textcircled{1}} \;:=\; -\frac{\Delta v_{\textcircled{1}}}{\Delta v_{\textcircled{1}*\textcircled{1}}} \cdot m^{(\text{inert})}_{\textcircled{1}} \quad .$$

The axiomatic system does not predict what the ratio of accelerations is - one has to make the experiment. If we would find

$$\frac{\Delta v_{\textcircled{1}}}{\Delta v_{\textcircled{1}*\textcircled{1}}} \;\stackrel{?}{=}\; -2$$

we could deduce according to hypothesis (1) for the inertial masses

$$m^{(\text{inert})}_{\textcircled{1}*\textcircled{1}} \;:=\; 2 \cdot m^{(\text{inert})}_{\textcircled{1}} \quad . \tag{2}$$

What guarantees the factor 2? In Classical Mechanics we measure $\Delta v_{\textcircled{1}}/\Delta v_{\textcircled{1}*\textcircled{1}} = -2$ while in Relativistic Dynamics in general we measure $\Delta v_{\textcircled{1}}/\Delta v_{\textcircled{1}*\textcircled{1}} \neq -2$. Similarly we can compose multiple objects $\underbrace{\textcircled{1}*\ldots*\textcircled{1}}_{N\times}$. Is the quantification of inertial mass in each case of empirical nature or is it a certain truth?

In Classical Mechanics we have one more notion of mass. The weight is quantified by means of a functioning beam scale [23] and a set of physically identical weight units $\{\textcircled{1}\}$. For an undetermined weight $\mathcal{A}$ on the left arm of the scale one successively adds weight units $\textcircled{1}$ to the right until the scale is in static equilibrium. The weight of $\mathcal{A}$ is quantified by counting the number of weight units $\sharp\{\textcircled{1}\}$ on the right side

$$m^{(\text{weight})}_{\mathcal{A}} \;:=\; \sharp\{\textcircled{1}\} \cdot m^{(\text{weight})}_{\textcircled{1}} \quad . \tag{3}$$

Here the weight for our composite body $\textcircled{1}*\textcircled{1}$ from above is unambiguously quantified

$$m^{(\text{weight})}_{\textcircled{1}*\textcircled{1}} \;:=\; 2 \cdot m^{(\text{weight})}_{\textcircled{1}} \quad .$$

The factor 2 is guaranteed by the *congruence principle.* Under the conditions of weight measurements all weight units ① are congruent to one another because they are *physically equivalent* and - when placed into the static(!) scale - they *behave in the same way.* The basic quantification of weight is constituted by counting those congruent units.

Our simple example illustrates that on the basis of Newton's Axiomatic System one cannot uniquely determine the dynamics of collision interactions without further implicit assumptions.[2] In particular the formalism lacks the congruence principle which is constitutive for basic measurement operations. We develop a measurement-theoretical foundation for the physical specification of interactions. It will entail not only the equations of motion from Newton's axiomatic formulation but also uncover the physical and methodical basis for unambiguous measurements of all actions in Classical and Relativistic Dynamics. In this way one can understand that quantity of 'inertial mass' with regard to interaction behavior coincides with the simple notion amount of matter. We uncover conditions (and limitations) for 'force' as meaningful derived physical quantity.

2 Foundation of Basic Physical Measurement

We are searching for a strictly physical foundation of Physics where initially Mathematics must remain outside - and then every step where Mathematics is introduced requires extra justification (by abstraction). Our problem is the measurement and thus the construction of physical measures. That is a question of *quantification and qualification.* Since - according to Hegel - a measure is the unity of quantity and quality (which physicists also name dimension). The English term 'quantification' constantly misleads to speak about numbers where in truth the talk is about measures. Our problem is called physical measurement (as unity of quantification and qualification) or 'On the construction of physical measures'.

Therein we encounter the following set of problems: What are those physical operations which we note down in theoretical physics simply by mathematical operation symbols. Is the '+' in 1kg + 1kg the same symbol as in 1sec + 1sec? Asking this question means to immediately recognize: No! The association of two weights is not the same as the connection of two durations etc., the concatenation of two lengths requires Euclidian Geometry as a physical theory, because the very operation requires that long objects are aligned against

[2]As a guiding principle in a search for deeper understanding Zeilinger [31] highly recommends 'to follow the guidance of the Copenhagen interpretation, that is, not to make any unnecessary assumptions not supported by a thorough analysis of what it really means to make an experiment.' Zeilinger regards as first step of a physical interpretation the analysis of 'rules that determine which element of the formalism corresponds to which measurable quantity or to which observable fact in a concrete experimental situation. These rules are a large, mostly not explicated but only implicit, set of instructions. They concern the instructions on how to proceed in experiment in order to demonstrate or test a theoretical prediction.' As instructive example of a well-founded theory he refers to Einstein's theory of Special Relativity: 'Almost all relativistic equations... of 1905 were known already before... But only Einstein created the conceptual foundations... *from which the equations* of the theory of relativity *arise.* He did this by introducing the principle of relativity, which asserts that the laws of physics must be the same in all inertial systems... together with the constancy of the velocity of light.'

each other in an angle of 180°. Such determination of a measurement operation makes of course no sense with regard to seconds. The next question is: What do physicists actually mean when they say $\mathbf{F} = \mathbf{p}/t$? Is an impulse meant to be divided by a duration? That is simply nonsense. There must be clarity what kind of (physical) operation the formation of that proportion is. Physicists combine quantities by 'multiplication' $\mathbf{p} = m \cdot \mathbf{v}$ which however is only explained if respective variables are meant to be numbers. How can we physically understand what a physicist does when he combines a mass m with a velocity $\mathbf{v}$? We can raise these questions for all mathematical operation symbols in physical equations - and only therewith we actually pose the foundation of Physics from solely physical grounds [20] as a general problem. This work concerns *physical objects* and the problem of determining *physical operations* really in a strictly physical way.

Physical theories ought to be built up from colloquial notions such as 'long bodies', 'energetic sources' etc. in terms of which one can define pre-theoretic ordering relations such as 'longer than', 'more energetic than'. One of the questions addressed in previous work is how to make the transition from these pre-theoretic notions to quantification, such as making the transition from the ordering relation of 'longer than' (if one body covers the other) to the notion of *how much* longer. In similar way we introduce dynamical notions 'more impulse' (if in a collision one object overruns the other) and 'more energetic' (if the effect of one source exceeds the effect of the other). In a physical model - built by coupling congruent standard actions - those basic pre-theoretic notions become measurable. We derive (classical and relativistic) equations between basic physical quantities of energy, momentum and inertial mass and ultimately the principle of least action.

Mathematical addition and physical 'addition' are different operations. Helmholtz [5] distinguishes the act of counting and measuring. The association of numbers $(2+1=3)$ is a different activity than the association of physical measures $(2m'+'1m'='3m)$. The physical meaning of 'quantity' and 'equal' is bound to the conduct of physical operations. We make a clear distinction between objects and operations. We begin with the problem of Kant and grasp all mathematical operation symbols which are used in theoretical physics as real actions. By *act of a physicist* we mean all tasks to carry out a basic physical measurement. This is not mathematical behavior; his operations are sensual concrete.

His *objects* are physical (not kitchen utensils or medical instruments), that means their *attributes* refer to physical behavior. In Dynamics they refer to the impulse and energetic behavior of objects in interacting systems. The *operations* which a physicist conducts to concatenate his measurement units are physical operations (not farming activities or medical conduct because the physicist is dealing with energies and with numbers). This *physical activity* is a particular kind of human activity. It is the way how physicists construct experimental apparatuses to make lengths and durations and energies and momenta measurable (i.e. to quantify them). There are different ways of associating physical objects which depend on the kind of basic measure (we connect meter sticks in different way than weights in a scale or than units of energy). Physicists conduct different kinds of association depending on the kind of basic measure which the respective measurement device (meter stick, watch, energetic unit resp. momentum unit) represents. This work is a contribution to understanding the active role of a physicist, his interventions in basic measurements.

On the basis of everyday work experience we roughly know what is meant by 'potential to cause action', 'striking power' and 'impulse'. These colloquial notions refer to physical behavior of objects in interactions. We introduce pre-theoretic *ordering relations* (of their energetic, impulse and inertial behavior) such that they are reproducible: (i) the comparison method is universally available and (ii) the act of comparison is intersubjectively interchangeable with regard to the individual observer. The measurement result is the same for every individual physicist. In the act of *quantification* we make the transition from basic dynamical notions 'more impulse' (if in a collision one object overruns the other) and 'more energetic' (if the effect of one source of energy exceeds the effect of the other) to the notion of *how much* more energetic etc.

We introduce basic physical measures (for energy, momentum and inertial mass) by a method which is itself of a physical nature (in the above sense):

1. Objects have to be of physical nature.
2. Qualities of those objects have to be of physical nature.
3. Measurability/metrizability of these attributes is achieved by the fact that
   (a) measurement devices are producible and reproducible in a physical way and
   (b) concatenation (Latin: association) of these measurement units has to be realized by physical operations.

Who would think that at the billiard table one acts as a physicist? We are dealing there with interactions of motion.[3] One begins to behave as a physicist if one sorts and couples these actions in a controlled way. We build a physical model - e.g. for the absorption action in a Calorimeter W_{cal} - by providing a standard action and by coupling these congruent standards in an organized way. In this model we can count the number of equivalent measurement devices. In this way those pre-theoretical notions 'energy' and 'momentum' in interactions become measurable - in an observer independent reproducible way. The congruence principle is constitutive for basic physical measurements. We uncover the origin of (basic) physical quantities of Energy, Momentum and Inertial Mass.

We derive all equations of (classical and relativistic) Dynamics without circular mathematical presupposition. 'In Gedanken' we can carry out actual measurements of energy and momentum in a fundamental manner - by controlling the coupling $*$ in a layout of solely congruent unit actions $w_{\mathbf{1}}$ we construct a physical model $W_{\text{cal}} := w_{\mathbf{1}} * \ldots * w_{\mathbf{1}}$ which reproduces the absorption action in a Calorimeter - and thus justify fundamental equations in (Classical and Relativistic) Dynamics based on physical and methodical principles. We derive the mathematical formalism of Dynamics from principles of empirical practice.

[3] We distinguish the mathematical expression 'action functional' (which physicists simply name 'action') form the empirical concept of a 'physical interaction' {2.2}.

Definition 1 *A basic physical measure is the quantification of a pre-theoretic ordering relation (with regard to one particular attribute of the physical behavior) of a physical object.*

Remark 1 *A physicist characterizes a physical object in a pre-theoretic ordering relation. In a basic measurement this characterization is quantified. He conducts (reproducible) physical operations with physical measurement devices: By concatenating (congruent) measurement units he constructs a material model which reproduces the physical object. With regard to the ordering relation both the object and the constructed physical model can substitute one another. In the material model he can count the number of (congruent) units. By means of this model the ordering relation becomes measurable. By this constructible substitution the pre-theoretic characterization of (the attribute of) the physical object is quantified.*

2.1 Outline

For illustration we review basic physical measurements of relativistic motion [24]. We render that order of co-existence and succession more precisely. The 'extent' of distances and durations and the 'form' of relative motion become measurable. We outline our method of metrization in the case of spatiotemporal ordering relations in Kinematics.

We begin with long *physical objects* $\mathcal{O}_l$ and enduring processes $\mathcal{P}_t$. On the basis of everyday experience we roughly know what is meant by 'length' and 'duration'. These *qualities* refer to their physical behavior. Two long objects (resp. two enduring processes) are comparable in a quantitative way. The *ordering relation* with regard to their attribute length $\sim_l$ (resp. duration $\sim_t$) is determined by the physical behavior of the two objects during the act of comparison. The way of comparison is physically specified so that it is reproducible in an observer independent way:

- $\sim_l$ if two extended objects lie on top of each other - one will *cover* the other
- $\sim_t$ if two processes begin simultaneously - one will *outlast* the other.

We quantify the 'how much' longer is one object than the other or provided two enduring processes the 'how much' more one of them lasts. By means of a material model this pre-theoretical ordering relation with regard to length and duration becomes measurable [21].

For reproducible measurements of length and duration we manufacture rulers and clocks. Those *measurement units* have to be universally reproducible. They provide sufficiently constant reference devices in comparisons of length resp. duration.[4] When $\mathcal{A}$lice conducts a physical measurement of the distance to $\mathcal{O}$tto the attribute 'physical' refers to operations by

[4] Measurement instruments are to be understood - not simply as arbitrary designations of natural objects but - as *artifacts*: Rulers and clocks are the product of norm realizing manufacturing actions [23]. To improve his manufacturing technique the producer judges the admissibility of his product (for the practical requirements of conducting measurements). He takes guidance in *test procedures*. They involve guidelines for the examination of the *straight form* of a constructed ruler and of the *uniform running* of a clock [22].

$\mathcal{A}$lice with her measurement units $\mathbf{1}_s$. She *concatenates* her rulers side by side in a straight way until the layout - symbolized by $\mathbf{1}_s * \ldots * \mathbf{1}_s$ - covers her path to $\mathcal{O}$tto $\overline{\mathcal{AO}}$

$$\mathbf{1}_s * \ldots * \mathbf{1}_s \quad =_s \quad \overline{\mathcal{AO}} \ .$$

$\mathcal{A}$lice quantifies the length of path $\overline{\mathcal{AO}}$ by constructing a material model. In the model she counts the number of connected units $\sharp \{\mathbf{1}_s\}$. By this *constructible substitution* the ordering relation 'length' becomes measurable. In this way $\mathcal{A}$lice relative distance to $\mathcal{O}$tto $s_{\overline{\mathcal{AO}}} = \sharp \{\mathbf{1}_s\} \cdot s_{\mathbf{1}}$ is *metricized* by the number of units and the length of her unit device $s_{\mathbf{1}}$.

In a physical measurement of relativistic motion we quantify spatiotemporal ordering relations. In laser ranging $\mathcal{A}$lice sends out light towards $\mathcal{O}$tto $\mathcal{A}_1 \rightsquigarrow \mathcal{O} \rightsquigarrow \mathcal{A}_2$ and towards $\mathcal{B}$ob $\mathcal{A}_1 \rightsquigarrow \mathcal{B} \rightsquigarrow \mathcal{A}_3$ and waits until their reflection returns. In radar round trips we focus on the distance traveled and $\mathcal{A}$lice waiting time. For a pair of light cycles $\mathcal{A}_1 \rightsquigarrow \mathcal{O} \rightsquigarrow \mathcal{A}_2$ and $\mathcal{A}_1 \rightsquigarrow \mathcal{B} \rightsquigarrow \mathcal{A}_3$ $\mathcal{A}$lice notices the order in which the light returns. She notes whether waiting time $t_{\overline{\mathcal{A}_1\mathcal{A}_2}}$ is larger than the other $t_{\overline{\mathcal{A}_1\mathcal{A}_3}}$. By the Light principle a longer waiting time for the independently moving light corresponds to a larger distance traveled $s_{\overline{\mathcal{AO}}}$ resp. $s_{\overline{\mathcal{AB}}}$ from $\mathcal{A}$lice to the turning point $\mathcal{O}$tto resp. $\mathcal{B}$ob and back. The physical process of propagating light in closed light cycles $\mathcal{A}_1 \rightsquigarrow \mathcal{O} \rightsquigarrow \mathcal{A}_2$ constitutes an *ordering relation* with regard to $\mathcal{A}$lice round trip waiting time.

For reproducible comparison of length resp. duration $\mathcal{A}$lice chooses a measurement unit. She constructs a *light clock* $\mathbf{1} : \mathcal{L}_I \rightsquigarrow \mathcal{L}_{II} \rightsquigarrow \mathcal{L}_I \ldots$ with two nearby mirrors $\mathcal{L}_I$ and $\mathcal{L}_{II}$ in a rigid frame (produced according to protophysical manufacturing norms). Between both mirrors the light constantly oscillates back and forth. The utilization of this measurement unit essentially refers to oscillating light inside. This physical process realizes the aspired ideal of uniform running more precisely. The motion of light becomes a measurement standard itself (classical light principle).[5]

With moving light we always find the length of its motion $s_{\mathbf{1}}$ (within the (ticking) light clock) together with its duration $t_{\mathbf{1}}$ (of (light clock) ticks). In practical measurements with light clocks $\mathbf{1}$ both moments of motion 'length' and 'duration' are always addressed unified.

Remark 2 *In a light clock both aspects length and duration are distinguishable but inseparably unified. Depending on how the measurement units* $\mathbf{1}$ *are physically concatenated* $*$

1. *adjacent way of connecting* $*_s$ *(ticking) light clocks represents a unit of distance* $\mathbf{1}_s$ *and*
2. *consecutive way of connecting* $*_t$ *(light clock) ticks represents a unit of duration* $\mathbf{1}_t$*.*

In classical laser ranging $\mathcal{A}$lice operates with closed light cycles to $\mathcal{O}$tto $\mathcal{A}_1 \rightsquigarrow \mathcal{O} \rightsquigarrow \mathcal{A}_2$ and also with the oscillating light inside her light clock $\mathbf{1}$. For the physical measurement

[5] In order to give physical meaning to the concept of time - Einstein demands - requires the use of some process which establishes relations between distant locations. In principle one could use any type of process. Most favorable for the theory one chooses a process about which we know something certain. For the free propagation of light this holds much more than for any other process [9].

of $\mathcal{O}$tto's relative motion she *successively connects* a swarm of neighboring light clocks by matching the independent propagation of light between them. Depending on the consecutive resp. adjacent way in which they are concatenated - the length of her *measurement unit* $\mathbf{1}$ represents *unit distance* $s_{\mathbf{1}}$ and each successive tick takes *unit time* $t_{\mathbf{1}}$. By means of physical concatenation $*$ of her light clocks $\mathbf{1}$ in consecutive and adjacent ways $\mathcal{A}$lice produces a material model. In the model she can count the number of congruent units $\mathbf{1}$. By means of this model the spatiotemporal ordering relations become measurable. By constructible substitution the pre-theoretical characterization of relative motion between $\mathcal{A}$lice and $\mathcal{O}$tto is metricized by the number of units, their layout and their length resp. duration.

Time-like Concatenation: $\mathcal{A}$lice joins together light clock ticks $\mathbf{1}$ *one after another* until the sequence - symbolized by $\mathbf{1} *_t \ldots *_t \mathbf{1}$ - reproduces the waiting interval of her laser ranging $\mathcal{A}_1 \rightsquigarrow \mathcal{B} \rightsquigarrow \mathcal{A}_2$

$$\overline{\mathcal{A}_1\mathcal{A}_2} =_t \mathbf{1} *_t \ldots *_t \mathbf{1} \tag{4}$$

with regard to duration $\sim_t$. Sequence of ticks and her waiting interval have equal duration $t_{\overline{\mathcal{A}_1\mathcal{A}_2}} = t_{\mathbf{1}*_t\ldots*_t\mathbf{1}}$. In sequence $\mathbf{1} *_t \ldots *_t \mathbf{1}$ $\mathcal{A}$lice concatenates light clocks $\mathbf{1}$ in a time-like way. Each measurement unit represents a unit of duration $\mathbf{1}_t$. In her material model $\mathcal{A}$lice counts the number of congruent ticks $\mathbf{1}_t$ - symbolized by $\sharp\,\{\mathbf{1}_t\} =: t^{(\mathcal{A})}_{\overline{\mathcal{A}_1\mathcal{A}_2}}$. By this constructible substitution the ordering relation 'duration' becomes measurable. In this basic measurement the duration of $\mathcal{A}$lice laser ranging interval

$$t_{\overline{\mathcal{A}_1\mathcal{A}_2}} \overset{(4)}{=} t_{\mathbf{1}*_t\ldots*_t\mathbf{1}} =: t^{(\mathcal{A})}_{\overline{\mathcal{A}_1\mathcal{A}_2}} \cdot t_{\mathbf{1}} \tag{5}$$

is metricized by the number of congruent units and the duration of her (light clock) tick $t_{\mathbf{1}}$.

Space-like Concatenation: Furthermore $\mathcal{A}$lice can join together ticking light clocks $\mathbf{1}$ *side by side*. $\mathcal{A}$lice utilizes the same *units* $\mathbf{1}$ to produce an adjacent layout of comoving light clocks. It represents $\mathcal{A}$lice *simultaneous straight measurement path* towards $\mathcal{O}$tto $\overline{\mathcal{A}\mathcal{O}}$. This layout - symbolized by $\mathbf{1} *_s \ldots *_s \mathbf{1}$ - reproduces the path of her laser ranging to $\mathcal{O}$tto

$$\overline{\mathcal{A}\mathcal{O}} =_s \mathbf{1} *_s \ldots *_s \mathbf{1} \ . \tag{6}$$

with regard to length $\sim_s$. Layout of light clocks and her laser ranging path have equal length $s_{\overline{\mathcal{A}\mathcal{O}}} = s_{\mathbf{1}*_s\ldots*_s\mathbf{1}}$. In layout $\mathbf{1}^{(1)} *_s \ldots *_s \mathbf{1}^{(n)}$ $\mathcal{A}$lice concatenates light clocks $\mathbf{1}^{(i)}$ in a space-like way. Each measurement unit represents a unit of length $\mathbf{1}_s$. In her material model $\mathcal{A}$lice counts the number of congruent clocks $\mathbf{1}_s$ - symbolized by $\sharp\left\{\mathbf{1}^{(i)}_s\right\} =: s^{(\mathcal{A})}_{\overline{\mathcal{A}\mathcal{O}}}$. By this constructible substitution the ordering relation 'length' becomes measurable. In this basic measurement the length along $\mathcal{A}$lice laser ranging path

$$s_{\overline{\mathcal{A}\mathcal{O}}} \overset{(6)}{=} s_{\mathbf{1}*_s\ldots*_s\mathbf{1}} =: s^{(\mathcal{A})}_{\overline{\mathcal{A}\mathcal{O}}} \cdot s_{\mathbf{1}} \tag{7}$$

is quantified by the number of congruent units and the length of her (ticking) light clock $s_{\mathbf{1}}$.

Spacetime-like Concatenation: In every single laser ranging $\mathcal{A}_1 \rightsquigarrow \mathcal{O} \rightsquigarrow \mathcal{A}_2$ $\mathcal{A}$lice determines the relative position of $\mathcal{O}$tto at moment $\mathcal{O}$ when her radar pulse reflects. $\mathcal{A}$lice joins her swarm of light clocks in both space-like and time-like way: She connects a consecutive sequence of (light clock) ticks until the 'half-time' moment $\mathcal{A}$

$$\overline{\mathcal{A}_1\mathcal{A}} \ =_t \ \mathbf{1} *_t \ldots *_t \mathbf{1}|_{\mathcal{A}}$$

in light clock $\mathbf{1}|_{\mathcal{A}} \equiv \mathbf{1}^{(1)}|_{\mathcal{A}}$ to an adjacent layout of (ticking) light clocks towards moment $\mathcal{O}$

$$\overline{\mathcal{A}\mathcal{O}} \ =_s \ \mathbf{1}^{(1)}|_{\mathcal{A}} *_s \ldots *_s \mathbf{1}^{(n)} \ .$$

The collective motion of light inside composite layout of ticking light clocks - symbolized by $\mathbf{1} *_t \ldots *_t \mathbf{1}^{(1)} *_s \ldots *_s \mathbf{1}^{(n)}$ - reproduces the motion of the outgoing light from $\mathcal{A}$lice to $\mathcal{O}$tto

$$\begin{aligned} \overline{\mathcal{A}_1\mathcal{O}} \ &=_{t,s} \ \mathbf{1} *_t \ldots *_t \mathbf{1}^{(1)} *_s \ldots *_s \mathbf{1}^{(n)} \\ &=: \ t^{(\mathcal{A})}_{\overline{\mathcal{A}_1\mathcal{A}}} \cdot \mathbf{1}_t \ *_s \ s^{(\mathcal{A})}_{\overline{\mathcal{A}\mathcal{O}}} \cdot \mathbf{1}_s \end{aligned} \tag{8}$$

with regard to spatiotemporal distance $\sim_{t,s}$. Configuration of light clocks and her laser ranging path cover equal spatiotemporal interval $(t,s)_{\overline{\mathcal{A}_1\mathcal{O}}} = (t,s)_{\mathbf{1}*_t\ldots*_t\mathbf{1}^{(1)}*_s\ldots*_s\mathbf{1}^{(n)}}$. In layout $\mathbf{1} *_t \ldots *_t \mathbf{1}^{(1)} *_s \ldots *_s \mathbf{1}^{(n)}$ $\mathcal{A}$lice utilizes the same light clocks $\mathbf{1}$ as spatiotemporal units. Along consecutive segment $\mathbf{1} *_t \ldots *_t \mathbf{1}^{(1)}$ each measurement unit represents a unit of time $\mathbf{1}_t$. Along adjacent segment $\mathbf{1}^{(1)} *_s \ldots *_s \mathbf{1}^{(n)}$ each light clock represents a unit of distance $\mathbf{1}_s$. In both segments of her material model $\mathcal{A}$lice counts the number of congruent ticks $\mathbf{1}_t$ - symbolized by $\sharp \{\mathbf{1}_t\} =: t^{(\mathcal{A})}_{\overline{\mathcal{A}_1\mathcal{B}}}$ - resp. number of congruent clocks $\mathbf{1}_s$ - symbolized by $\sharp \left\{\mathbf{1}^{(i)}_s\right\} =: s^{(\mathcal{A})}_{\overline{\mathcal{A}_1\mathcal{O}}}$. By this constructible substitution the ordering relation 'spatiotemporal distance' becomes measurable. In this basic measurement the spatiotemporal distance from $\mathcal{A}$lice towards $\mathcal{O}$tto

$$\begin{aligned} (t,s)_{\overline{\mathcal{A}_1\mathcal{O}}} \ &\overset{(8)}{=} \ (t,s)_{\mathbf{1}*_t\ldots*_t\mathbf{1}^{(1)}*_s\ldots*_s\mathbf{1}^{(n)}} \\ &\overset{(5)(7)}{=} \ \left(t^{(\mathcal{A})}_{\overline{\mathcal{A}_1\mathcal{A}}} \cdot t_{\mathbf{1}} \ , \ s^{(\mathcal{A})}_{\overline{\mathcal{A}\mathcal{O}}} \cdot s_{\mathbf{1}} \right) \ = \ \left(\frac{1}{2} \cdot t^{(\mathcal{A})}_{\overline{\mathcal{A}_1\mathcal{A}_2}} \cdot t_{\mathbf{1}} \ , \ t^{(\mathcal{A})}_{\overline{\mathcal{A}_1\mathcal{A}_2}} \cdot s_{\mathbf{1}} \right) \end{aligned} \tag{9}$$

is metricized by the number of congruent units, the consecutive or adjacent way of their connection and the duration of her (light clock) tick $t_{\mathbf{1}}$ and length of (ticking) light clock $s_{\mathbf{1}}$. By repeated laser ranging measurements with closed light cycles $\mathcal{A}$lice obtains the unified spatiotemporal measure of $\mathcal{O}$tto's relative motion.

$\mathcal{A}$lice characterizes $\mathcal{O}$tto's relative motion with a laser ranging technique. She probes the spatiotemporal order of distant objects by a physical process: the propagation of light. Each radar round trip characterizes the distance traveled and her waiting time. $\mathcal{A}$lice specifies the spatiotemporal order of demonstrable objects in colloquial language. This characterization of $\mathcal{O}$tto's relative motion is quantified in classical measurements with light clocks. $\mathcal{A}$lice constructs light clocks $\mathbf{1}$ in a physical way. She concatenates those measurement units by an - observer independent - physical process: She successively connects a swarm of neighboring light clocks by matching the independent propagation of light between them and lays them out in a consecutive and adjacent way. $\mathcal{A}$lice builds assemblies of long layouts of (ticking) light clocks $\mathbf{1} *_s \ldots *_s \mathbf{1}$ and enduring sequences of (light clock) ticks $\mathbf{1} *_t \ldots *_t \mathbf{1}$.

In laser ranging practice with light clocks $\mathbf{1}$ $\mathcal{A}$lice never deals with just one light ray; operations with her measurement unit always involve both outgoing and returning pulse. In the quantification of laser ranging we solely deal with two-way light cycles: (i) inside individual light clocks $\mathbf{1}$ and (ii) in suitably connected configurations of light clocks. The measurement principle is that: Given the *elementary* two-way light cycle in measurement unit $\mathbf{1}$ we analyze in measurement practice complex configurations of those elementary two-way light cycles in suitably constructed layouts of ticking light clocks $\mathbf{1} *_t \ldots *_t \mathbf{1} *_s \ldots *_s \mathbf{1}$.

Remark 3 *The meaning of basic physical measures arises - not by chopping measurement units* $\mathbf{1}$ *into pieces but instead - by concatenating many congruent measurement units* $\mathbf{1}$ *(each taken as an inseparable unity) to construct material models* $\mathbf{1} * \cdots * \mathbf{1}$.

Remark 4 *The method of a basic measurement involves a doubling of physical measures. Both the measurement object and the material model represent physical objects. We compare the physical behavior in both processes: The collective motion of light inside* $\mathcal{A}$*lice layout of ticking light clocks reproduces the relative motion of* $\mathcal{O}$*tto with regard spatiotemporal order. Her material model contains solely congruent kinematical units* $\mathbf{1}$ *and it is (locally) invariant under permuting their order. In that model* $\mathcal{A}$*lice counts the number of congruent light clocks. By means of this model the spatiotemporal ordering relation becomes measurable; the spatiotemporal interval of* $\mathcal{O}$*tto's relative motion is metricized.*

We can justify relations between basic physical measures: $(t,s)_{\overline{\mathcal{A}_1\mathcal{A}}} + (t,s)_{\overline{\mathcal{A}\mathcal{O}}} = (t,s)_{\overline{\mathcal{A}_1\mathcal{O}}}$. The quantity in each term of the equation is the number of light clocks concatenated in underlying material model $\mathbf{1} *_t \ldots *_t \mathbf{1}^{(1)}$ resp. $\mathbf{1}^{(1)} *_s \ldots *_s \mathbf{1}^{(n)}$. Physicists assemble these kinematical units $\mathbf{1}$ by physical operations - matching the independent propagating of light between them. The arithmetic addition of the number of light clocks for composite layout $\mathbf{1} *_t \ldots *_t \mathbf{1}^{(1)} *_s \ldots *_s \mathbf{1}^{(n)}$ is a genetic consequence of measurement operations.

In elementary laser ranging from $\mathcal{A}$lice towards $\mathcal{O}$tto $\mathcal{A}_1 \rightsquigarrow \mathcal{O} \rightsquigarrow \mathcal{A}_2$ we distinguish her laser ranging path $\overline{\mathcal{A}\mathcal{O}}$ from the length of that route $s_{\overline{\mathcal{A}\mathcal{O}}}$. Similarly we distinguish her laser ranging interval $\overline{\mathcal{A}_1\mathcal{A}_2}$ from the duration of that interval $t_{\overline{\mathcal{A}_1\mathcal{A}_2}}$. $\mathcal{A}$lice characterizes object $\mathcal{O}$tto - as a physicist - with regard to an ordering relation $\sim_m$. That characterization of the object $\mathcal{O}$ is quantified in a basic measurement. The measure $m_{\mathcal{O}}$ is a quantification of that quality m of object $\mathcal{O}$. We distinguish measurement object $\mathcal{O}_m$ and its measure $m_{\mathcal{O}}$ - without separating them: For physical measure $m_{\mathcal{O}}$ we adhere to its extensional determination $\mathcal{O}$. The measure is a unity of quantity and quality.

Measures duration $t_{\mathcal{O}}$ resp. length $s_{\mathcal{O}}$ are qualities of physical process 'moving $\mathcal{O}$tto' [32] and as such they are shown and compared by means of the latter. A basic measurement appears as a pair comparison between measurement object $\mathcal{O}$tto and material model $\mathbf{1} * \cdots * \mathbf{1}$. Both have equal measure length (up to the practically sufficiently small *measurement error*)

$$s_{\mathcal{O}} \ = \ s_{\mathbf{1}*\cdots*\mathbf{1}} + \Delta s$$

if and only if both objects are interchangeable $\mathcal{O} \sim_s \mathbf{1} * \cdots * \mathbf{1}$ with regard to equivalence relation length. The measured value $s_{\mathcal{O}}^{(\mathcal{A})}$ in $\mathcal{A}$lice physical measure $s_{\mathcal{O}} \overset{(7)}{=} s_{\mathbf{1}*\ldots*\mathbf{1}} =: s_{\mathcal{O}}^{(\mathcal{A})} \cdot s_{\mathbf{1}}$ refers to measurement actions which $\mathcal{A}$lice must execute: For reproducible comparison of measure m she manufactures a suitable measurement unit $\mathbf{1}$. She concatenates congruent units to constructs a material model $\mathbf{1} * \ldots * \mathbf{1}$ which reproduces measurement object $\mathcal{O}$ with regard to ordering relation m. Both acts in $\mathcal{A}$lice measurement - constructing the material model and implementing comparison with regard to length - are intersubjectively reproducible. Therein lies the practical purpose and meaning of her basic physical measure.

The measurement theoretical foundation of Kinematics [24] has been done by protophysical considerations [22] in a form theoretical way and in particular without presupposing any formulation for equations of motion. Thus (Relativistic) Kinematics can be assumed as known and given for a circularity free foundation of Dynamics.

We begin the measurement theoretical foundation of Dynamics from our knowledge about interactions of motion which we have acquired from work experience in everyday life: In systems of interacting objects we characterize their physical behavior with regard to energy and momentum {2.2}. These characterizations - of interacting objects - are comparable in a quantitative way. The ordering relation with regard to energy $\sim_E$ and momentum $\sim_{\mathbf{p}}$ is determined by the physical behavior of the objects during the act of comparison. The way of comparison is physically specified so that it is reproducible in an observer independent way {2.3}. By basic dynamical measurements (similar as in Kinematics) these pre-theoretical ordering relations for energy and momentum $\sim_{E,\mathbf{p}}$ are quantified {3.1}.

2.2 Interaction of Motion

In theoretical physics the term 'action' is occupied by the action functional (which physicists simply name action). Only the expression 'physical interaction' makes unambiguous reference to the real process between natural objects.[6] There are many types of interactions: In chemical interactions (called 'reaction') the chemical bond between atoms is changing. Quantum mechanical interactions involve changes of internal degrees of freedom as spin or particle generation. The collision on a billiard table is an example for a physical interaction. The identity of interacting objects is preserved, only their state of motion changes. In the tradition of Euler we call them 'interaction of motion' (German: Bewegungswirkung) [2]. They include contact interactions as billiard, gravitational interactions in the solar system or electromagnetic interactions between charged particles. We specify initial and final state of respective particles by their energetic and impulse behavior. We determine these interactions in a physical way. Therefore they are called physical interactions. In the following we

[6]In the first case 'action' is an attribute of a functional. The action plays a subordinate role. It specifies a mathematical functional. In the second case 'physical' is the attribute of an interaction or simply an action ('inter' only emphasizes the fact that natural objects act against one another when they are in a system). We specify the interaction of natural objects by their physical behavior. From that empirical basis we begin our strictly physical foundation of Dynamics.

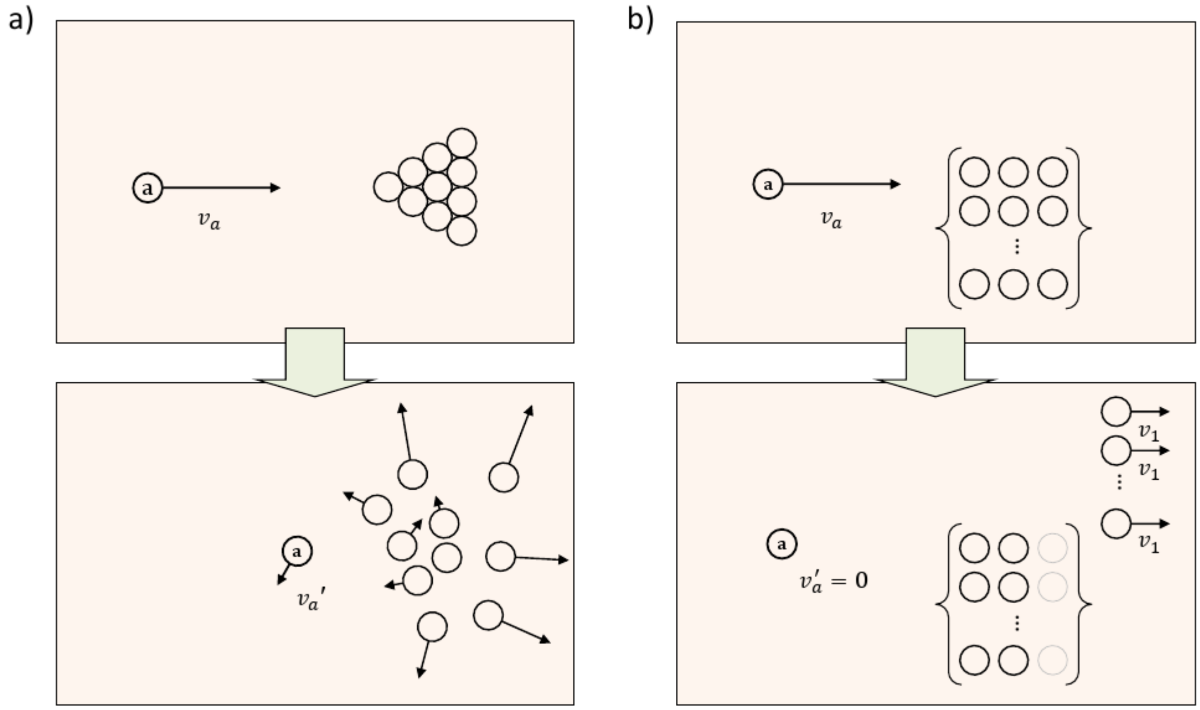

Figure 2: initial and final state of motion a) in generic billiard collision b) in controlled replacement process with calorimeter reservoir $\{①_{\mathbf{v}=\mathbf{0}}\}$

use the term interaction or short action strictly in this empirical sense.

We are referring to domains of everyday work experience where conditions for meaningful colloquial denominations 'material body' and 'motion' are practically sufficiently satisfied [15] [28]. We grasp an action impartially as the collective behavior of an interacting system. According to the Principle of Inertia a free object moves on his own. Its state of motion is preserved until it is affected by an external cause [2]. During an interaction of motion objects in system $\{G_1 \cup \ldots \cup G_N\}$ act against one another. Under the effect of the external cause each object acts against changes in its state of motion. After an interaction of motion those objects preserve their identity and solely changed their respective state of motion $\mathbf{v}_i$.

We illustrate the interventions of a physicist in basic physical measurements. This is not mathematical behavior; his operations are sensual concrete. Who would think that at the billiard table one acts as a physicist? We are dealing there with interactions of motion. One begins to behave as a physicist if one sorts and couples these actions in a controlled way.

When we step into a billiard room we deal with physical objects and their physical behavior. Moving billiard balls change their mutual position. Throughout their relative motion these objects preserve their character of a physical body. After each collision they remain solid rigid bodies and they keep their identity if furnished with individual colors and numbers. In collisions they solely undergo changes in their state of motion. According to the Principle of Inertia [2] free objects move on their own. Their state of motion is preserved until they hit the cushions or another ball. In a collision the balls act against one another. Under the effect of the external cause each ball acts against changes in their state of motion. The state of motion is preserved in the absence of an external cause. Dynamics explains changes in the state of motion due to external causes. After each collision balls fly apart from one another with changed motion and impulses. The potential momentum of each moving ball is realized against the next external obstacle which happens to come into the way.

The player executes the first break shot with the cue ball against object balls which are racked tightly and initially at rest (see figure 2a). After a generic collision the object balls fly off into arbitrary directions and each with arbitrary momentum. In the impact the moving cue ball expends its potential of causing actions and eventually comes to rest. The cue ball transfers its impulse onto the object balls which fly off in an uncontrolled way. We make the experience that a heavier and faster cue ball will cause more impact on the object balls than a slower would. We characterize the physical behavior of those objects by ordering relations. In this way we begin the physical approach to the facts of (billiard) actions. Conditions for comparing the physical behavior of these objects in a reproducible way are specified in {2.3}. The 'how much' more potential of action a player expends and 'how much' more impetus a cue ball carries is metricized in {3.1}.

When the player strikes he is dealing with interactions of motion. The player begins to behave as a physicist by the act of *controlling* the process and by *sorting* these actions of motion. This is physical behavior because the physicist is dealing with energies and with numbers. This physical activity is a particular kind of human activity. It is the way how physicists construct experimental instruments to make energy and momentum measurable. We construct a Gedanken-model on the operation of a particle detector: a calorimeter (see figure 3 with the process inside the calorimeter unspecified as 'black box'). It contains a reservoir with physically equivalent particles $\{\textcircled{1}_{\mathbf{v=0}}\}$ which are initially at rest and with which something will happen. An incoming particle $\textcircled{a}_{\mathbf{v}_a}$ will be slowed down $\textcircled{a}_{\mathbf{v}_a=0}$ in a cascade of successive collisions with those resting particles of the reservoir. Instead of a generic collision we set up a cascade of collisions where the process is controlled by a collective of physicists (see figure 2b). As a result a certain number of excited particles $\sharp\,\{\textcircled{1}_{\mathbf{v_1}}\}$ (with standardized velocity $\mathbf{v_1}$) will be knocked out of the calorimeter as well as a number of energy carriers (imagine 'batteries') which we symbolize $\sharp\left\{\mathbf{1}_E\big|_{\mathbf{v}}\right\}$.

This replacement process reproduces the effect of absorbing incident cue ball by object balls insofar as the incident cue ball finally comes to rest. In the state of rest it has expended its potential to cause further actions and its impetus. While in a generic billiard collision all object balls fly off with arbitrary momentum now the process in the cascade of collisions is set up and controlled. As a result - of absorbing incident cue ball - the object balls fly off in an organized way. Each ball represents a unit of momentum and each behaves in the same way. In this process we can count the number of congruent dynamical units. In this way dynamical ordering relations with regard to energy and momentum become measurable. By means of this physical model the energy and momentum of incident cue ball is metricized.

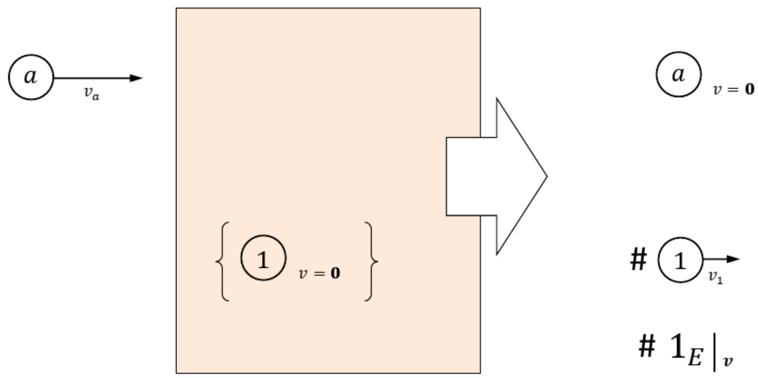

Figure 3: action in calorimeter

The condition for attaining basic physical measures is the existence of a collective of physicists. Implementation of the underlying replacement process requires to control the preparation and coupling of many unit actions $\{3.1\}$. The orchestration of many assistants belongs to behaving as a physicist. E.g. in figure 7 the coordinated effort of a team of assistants is required to set up and control a sequence of transversal collisions: Six men have to line up at the corners and obey instructions to timely operations in the right way in order to realize the desired progression for their replacement process. Physicists have to cooperate to construct material models. Everybody has to know *when* and *where* to pick the next initially resting object $\textcircled{1}$ from the reservoir of dynamical measurement devices $\{\textcircled{1}_{\mathbf{v}=\mathbf{0}}\}$ and *how* to catapult it into the way of the cue ball $\textcircled{a}_{\mathbf{v}_a}$ to achieve the desired result. The cooperation of physicists is community-building. The production collective is a social condition and belongs to the quantification of physical measures. Basic physical quantities are a joint product and not generated individually.

For the formulation of similar Gedanken-experiments we introduce following **notation**: We symbolize the 'physicality' of particles by $\textcircled{a}$ with placeholder a for individual names. In graphical representations we symbolize:

- $\textcircled{a}\xrightarrow[\mathbf{v}_a]{}$ for moving particles resp.
- $\textcircled{a}_{\mathbf{v}_a=0}$ or simply $\textcircled{a}_{\mathbf{0}}$ when the same object rests.

For the ongoing text and in mathematical equations we formulate:

- $\textcircled{a}_{\mathbf{x}_a,\mathbf{v}_a}$ for an object $\textcircled{a}$ at the location $\mathbf{x}_a$ and in the state of motion $\mathbf{v}_a$ or simply
- $\textcircled{a}_{\mathbf{v}_a}$ when the location does not matter (analogous $\mathbf{1}_E\big|_{\mathbf{v}}$ for moving sources of energy, e.g. tightened spring, charged crossbow, charged battery etc.) and finally
- $\textcircled{a}_{\mathbf{v}_a=0}$ or $\textcircled{a}_{\mathbf{0}}$ resp. $\mathbf{1}_E\big|_{\mathbf{0}}$ when the respective object rests.

2.3 Pre-theoretic Ordering Relation

We specify the physical determination of the comparison - of energy and momentum - such that it is reproducible:

- the comparison method is universally available and
- the act of comparison is intersubjectively interchangeable with regard to the individual observer [22].

A moving billiard ball has the potential to cause an action; a resting one does not. In the example of our billiard actions we execute the first break shot with the cue ball against object balls which are racked tightly and initially at rest. Moving cue ball expends its impetus and its potential to cause further actions and eventually comes to rest. The object balls fly off in arbitrary way. When we execute the break shot twice in different ways we can compare the 'striking power' of the cue balls in their respective effect on the object balls. We can also compare the 'impetus' of the two cue balls directly: In a collision where both initially move against one another into opposite directions the one of the two balls has more momentum which overruns the other.

We have two objects which are of the same kind (German: gleichartig). According to the Principle of Inertia both balls move on their own. Their state of motion is preserved unless they are affected by an external cause. In the collision each ball acts - under the influence of the external cause (the balls are impenetrable) - against changes of their state of motion.

Definition 2 *<u>Momentum</u> $\mathbf{P}_{\textcircled{a}_v}$ is the Impulse or Striking Power $\mathbf{P}$ of a moving object $\textcircled{a}_{\mathbf{v}_a}$ [4]. We compare Momentum in an interaction according to ordering relation $\sim_{\mathbf{P}}$: Object $\textcircled{a}_{\mathbf{v}_a}$ is more smashing than object $\textcircled{b}_{\mathbf{v}_b}$*

$$\textcircled{a}_{\mathbf{v}_a} \quad >_{\mathbf{P}} \quad \textcircled{b}_{\mathbf{v}_b} \tag{10}$$

if object $\textcircled{a}_{\mathbf{v}_a}$ <u>overruns</u> object $\textcircled{b}_{\mathbf{v}_b}$ - in an against one another directed collision.[7] Then the abstract Momentum of $\textcircled{a}_{\mathbf{v}_a}$ is larger than the Momentum of $\textcircled{b}_{\mathbf{v}_b}$

$$\mathbf{p}_{\textcircled{a}\,\mathbf{v}_a} \quad > \quad \mathbf{p}_{\textcircled{b}\,\mathbf{v}_b} \quad . \tag{11}$$

We compare both objects in this regard: both objects have <u>equal</u> momentum $\textcircled{a}_{\mathbf{v}_a} \sim_{\mathbf{P}} \textcircled{b}_{\mathbf{v}_b}$ if none of the two balls overruns the other in a collision. This *impulse behavior* is in both objects the same if and only if they are interchangeable in this practical action. The ordering relation with regard to 'striking power' is determined by the physical behavior of the objects during the act of comparison. With regard to striking power both objects $\textcircled{a}_{\mathbf{v}_a}$, $\textcircled{b}_{\mathbf{v}_b}$ have the same dynamical behavior. That quality (German: Art) 'impulse' is <u>identical</u> in both objects. Both objects have *identical impulse* $\mathbf{p}_{\textcircled{a}\,\mathbf{v}_a} = \mathbf{p}_{\textcircled{b}\,\mathbf{v}_b}$.

[7] Initially the two objects run into opposite directions.

Definition 3 *Inertia* $m^{(\text{inert})}_{ⓐ_v}$ *is the resistance* $m^{(\text{inert})}$ *of an object* $ⓐ_{\mathbf{v}_a}$ *against changes in its state of motion when under the influence of external causes [15] [2]. We compare Inertia in an interaction according to ordering relation* $\sim_{m^{(\text{inert})}}$*: According to Galilei object* $ⓐ_{\mathbf{v}_a}$ *has more inertial mass than object* $ⓑ_{\mathbf{v}_b}$

$$ⓐ_{\mathbf{v}_a} \quad >_{m^{(\text{inert})}} \quad ⓑ_{\mathbf{v}_b} \tag{12}$$

if object $ⓐ_{\mathbf{v}_a}$ *overruns object* $ⓑ_{\mathbf{v}_b}$ *- in a collision where initially both objects move against one another with same velocity* $\mathbf{v}_a \stackrel{!}{=} -\mathbf{v}_b$ *[10]. Then inertial mass of* $ⓐ_{\mathbf{v}_a}$ *is larger than the inertial mass of* $ⓑ_{\mathbf{v}_b}$

$$m^{(\text{inert})}_{ⓐ_{\mathbf{v}_a}} \quad > \quad m^{(\text{inert})}_{ⓑ_{\mathbf{v}_b}} \quad . \tag{13}$$

Remark 5 *The Inertia equivalence relation is a special case of the Momentum equivalence relation with an extra condition that both objects initially move with same velocity against one another* $\mathbf{v}_a \stackrel{!}{=} -\mathbf{v}_b$

$$\sim_{m^{(\text{inert})}} \quad := \quad \sim_{\mathbf{P}} \big|_{\mathbf{v}_a = -\mathbf{v}_b}$$

Consider a bow $\mathcal{B}$ and a crossbow $\mathcal{C}$. When the string is tightened (and mechanically locked) charged bow $\mathcal{B}_E$ and charged crossbow $\mathcal{C}_E$ become possible causes of actions. In the charged state each is a source of energy. They can realize their potential to cause an action if they are coupled into a system $\{\mathcal{G}_1 \cup \mathcal{G}_2\}$ of initially resting archer $\mathcal{G}_1$ and arrow $\mathcal{G}_2$ (see figure 4). We compare both energetic systems by comparison of their efficacy. According to the Equipollence Principle we compare the energy E of two causes of action $\mathcal{B}_E$ and $\mathcal{C}_E$ by means of their potential effect on the same system $\{\mathcal{G}_1 \cup \mathcal{G}_2\}$. The charged crossbow $\mathcal{C}_E$ is more energetic than the charged bow $\mathcal{B}_E$ if the first causes a larger change in the state of motion of archer v_1 and arrow v_2 than the latter would if coupled into the same system $\{\mathcal{G}_1 \cup \mathcal{G}_2\}$. In each case the *energetic cause* (charged crossbow $\mathcal{C}_E\big|_{\mathbf{0}}$ resp. charged bow $\mathcal{B}_E\big|_{\mathbf{0}}$) is initially at rest and - after expending the energy onto archer $\mathcal{G}_1$ and arrow $\mathcal{G}_2$ - both *discharged sources* (crossbow $\mathcal{C}$ resp. bow $\mathcal{B}$) remain at the state of rest.

Definition 4 *Energy* $E_{\mathcal{U}}$ *is the potential* E *of a cause of action* $\mathcal{U}$ *to effect a system* $\{\mathcal{G}\}$*. We specify the Energy of two possible causes of action* $\mathcal{U}_E$ *and* $\tilde{\mathcal{U}}_E$ *in a physical comparison according to equivalence relation* $\sim_E$*: Cause of action* $\mathcal{U}_E$ *is more energetic than cause of action* $\tilde{\mathcal{U}}_E$

$$\mathcal{U}_E \quad >_E \quad \tilde{\mathcal{U}}_E \tag{14}$$

if and only if the potential effect of cause $\mathcal{U}_E$ *on a system* $\{\mathcal{G}\}$ *exceeds the potential effect of cause* $\tilde{\mathcal{U}}_E$ *- if coupled into the same system* $\{\mathcal{G}\}$ *(Leibniz: Equipollence Principle see* $\{3.4.1\}$*). Then the abstract Energy of* $\mathcal{U}_E$ *is larger than the Energy of* $\tilde{\mathcal{U}}_E$

$$E_{\mathcal{U}} \quad > \quad E_{\tilde{\mathcal{U}}} \quad . \tag{15}$$

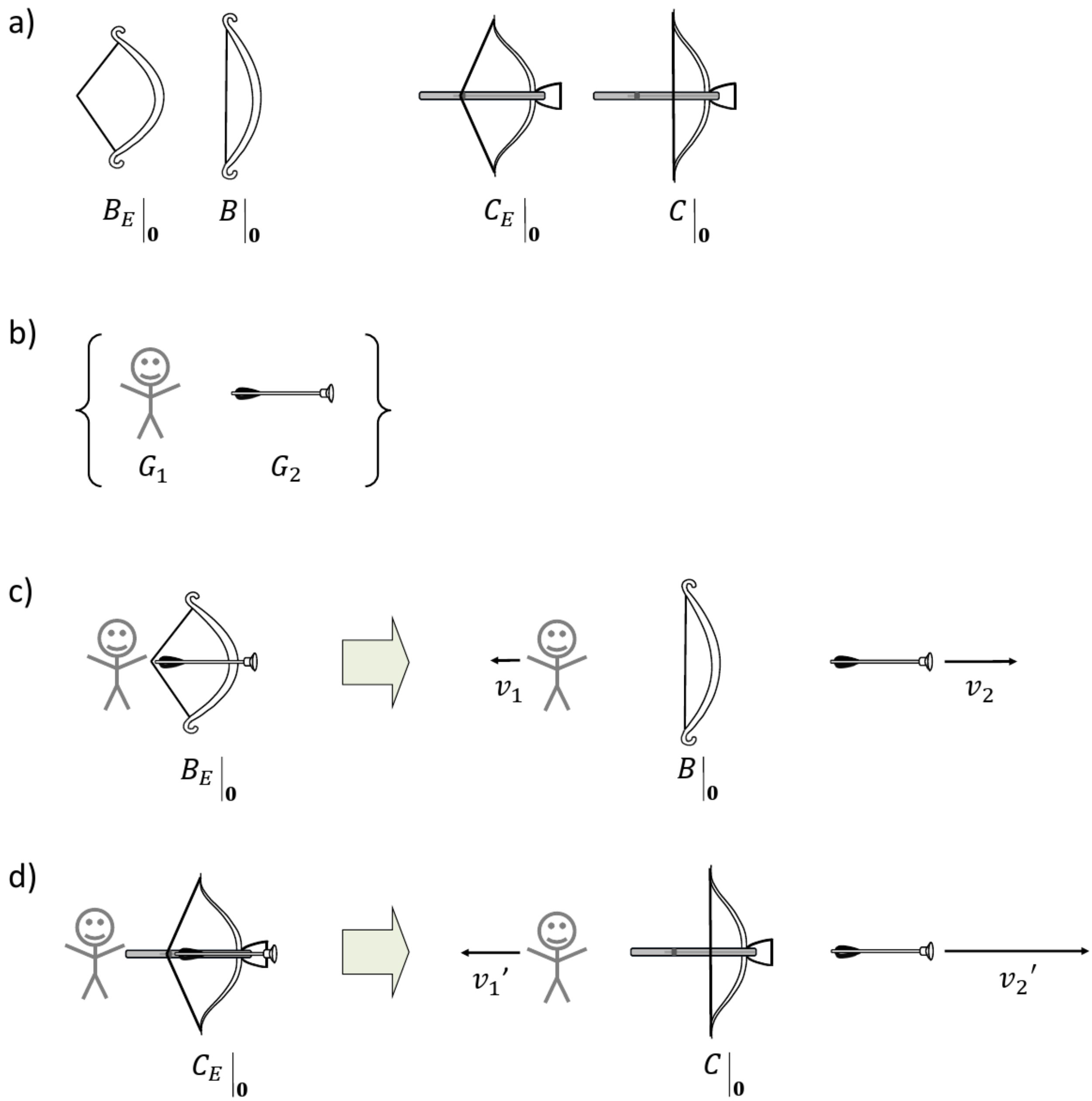

Figure 4: a) charged bow $\mathcal{B}_E$ and charged crossbow $\mathcal{C}_E$ (locked and at rest) are potential causes of action when coupled into b) system consisting of resting archer and arrow $\{\mathcal{G}_1 \cup \mathcal{G}_2\}$ c) charged bow $\mathcal{B}_E$ expends potential of action on system $\{\mathcal{G}_1 \cup \mathcal{G}_2\}$ (discharged bow $\mathcal{B}\big|_{\mathbf{0}}$ rests) d) likewise charged crossbow $\mathcal{C}_E$ realizes potential effect onto same system $\{\mathcal{G}_1 \cup \mathcal{G}_2\}$

We specify autonomous methods for comparing 'potential to cause actions' resp. 'striking power' without theoretical anticipation of motion as mathematical mapping $\gamma : \tau \mapsto (t, \mathbf{x})$. This would be an unwanted reduction of (interactions of) motion as a mathematical map. We grasp an action impartially as the collective behavior of an interacting system {2.2}. During an interaction of motion objects solely undergo changes in their state of motion. We treat each action as inseparable unity. The action is the atom in basic dynamical measurements! We conceptualize the physical specification of interactions of motion as a comparison of different actions. We compare the collective behavior in one interacting system with the collective behavior in another interacting system - with regard to energy and momentum. In this respect Physics goes beyond the scope of Mathematics: Mathematics already postulates given Abstracta whereas our focus concerns their formation (and prerequisites). We illustrate the act of a physicist in his empirical practice. We have introduced the comparison of energy resp. momentum circularity free - without presupposing prior quantification of basic dynamical measures and without anticipating any formulation for a dynamical equation of motion or for conserved quantities.

Basic physical measures 'energy' and 'momentum' are abstract. They are formed by comparing the collective behavior of interactions of motion in one particular regard. Interactions have a variety of *possible* appearances. Under the *condition* of our comparison procedures a singular aspect is distinguished for observation. It is their *common quality* $\mathbf{p}$, $m^{(\text{inert})}$ or E with regard to which two objects (interactions of motion) can substitute one another in presented comparison methods.

Remark 6 *In an abstraction we regard the common quality of both objects for itself without needing to consider the dissimilarity (of both objects in other regards).*[8]

Under the abstraction 'energy' resp. 'impulse' we regard individual objects (charged bow, charged crossbow, battery, moving billiard ball etc.) solely as substitutable representatives of their common quality 'potential to cause action' resp. 'striking power'. In the theory we make propositions about abstract 'energy' and 'momentum'. This transition is implemented by a *limitation in the manner of speaking* onto invariant assertions [13]. It is the transition from simple descriptive sentences (about individual interactions) onto such assertions which remain unchangedly valid under substituting equivalent representatives of quality E and $\mathbf{p}$. We restrict to assertions about interacting systems which remain valid if we substitute given source of energy resp. momentum by another source which is equivalent under specified

[8]Helmholtz [5] explains 'bodies whose weight we are comparing can be made from most different materials, different shape and volume. The weight - which we set equal - is only one of their attributes and obtained by abstraction. We are only justified - to call those bodies themselves weights and designate these weights as quantities - in circumstances where we can disregard all other properties of these bodies'.

Ruben [14] thinks about: 'a tree e.g. is in general a subject of Biology. If the tree is cut down then it is - for the worker who has to get out of the way of the falling tree - a mechanical object. In this context the tree is essentially important as carrier of weight; it is unimportant whether the tree is a linden or an oak. All natural things are always also carrier of mass. Insofar as they are they are subject of Mechanics.'

physical methods of comparison $\sim_{E,\mathbf{p}}$.[9]

3 Basic Dynamical Measures

We compare the - energetic and momentum - behavior of interacting objects by means of pre-theoretic ordering relations. They are familiar from everyday work experience. We explain the way of comparison at first in words or by examples. Physicists specify the act of comparison so that it is universally reproducible in an observer independent way. Those pre-theoretic notions are quantified in basic measurements. We give precise definition for the quantification of interactions of motion. We introduce the *terminology*, operational *denominations* and corresponding mathematical *formulation*. We substantiate the *formalism* of Classical Mechanics on the basis of pre-theoretic notions known from everyday work. In this way one can understand the relation of abstract mathematics to reality. We justify the validity of mathematical terms and quantitative equations by our life-world based definition of measurement termini. We investigate the question of the foundation of a non-mathematical science which however uses mathematics. Physics appears as the mother of its Mathematics in empirical practice [15]. This work is a contribution to understanding the active role of a physicist in basic measurements.

3.1 Quantification

So far (our knowledge acquired from work experience about) an interaction of motion w is characterized only in colloquial language by pre-theoretical ordering relations for energy $\sim_E$ and momentum $\sim_{\mathbf{p}}$. For the physical specification of action w:

1. We provide - as dynamical measurement device - a unit action '$w_{\mathbf{1}}$' which represents measurement units for energy '$\mathbf{1}_E$' and momentum '$\mathbf{1}_{\mathbf{p}}$'.

2. We introduce the operation for their physical concatenation '$*$'.

3. In a basic dynamical measurement (a collective of) physicists construct a material model '$w_{\mathbf{1}} * \ldots * w_{\mathbf{1}}$' by setting up and controlling the process in a sequence of coupled unit actions '$w_{\mathbf{1}}$'.

This physical model reversibly reproduces our action of motion w

$$w \quad \sim_{E,\mathbf{p}} \quad w_{\mathbf{1}} * \ldots * w_{\mathbf{1}} \tag{16}$$

[9] For example the descriptive sentence 'The archer has a cute crossbow with a steel bow' is an accurate description of the facts portrayed in figure 4d. When we replace the crossbow by a bow that assertion will lose its validity. Although the bow is cute too it is not made of steel but of wood. In contrast assertions about 'the recoil behavior of archer and arrow' remain valid when we replace the charged crossbow $\mathcal{C}_E$ by an equally charged bow $\mathcal{B}_E$ (see figure 4c) - despite differences in their inner dynamics in which crossbow $\mathcal{C}$ resp. bow $\mathcal{B}$ expend their energy and despite the fact that the bow is manufactured from wood.

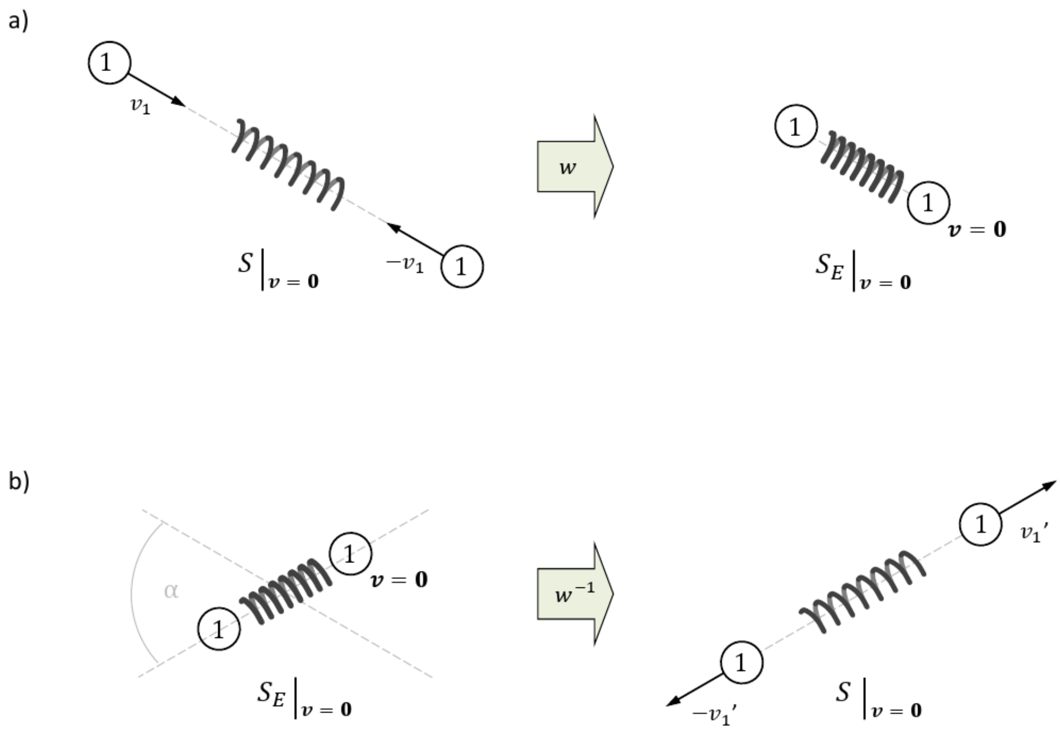

Figure 5: reversible action w by compression and decompression of resting spring a) initially neutral spring $\mathcal{S}\big|_{\mathbf{0}}$ gets compressed while two objects moving with diametrically opposed unit velocity come to rest b) reorientation and reversible relaxation of initially charged spring $\mathcal{S}_E\big|_{\mathbf{0}}$ with potential of action E boosts two resting objects with diametrically opposed unit velocity; in final state discharged source of energy (neutral spring $\mathcal{S}\big|_{\mathbf{0}}$) is again at rest

with regard to the energy- and momentum-characterization. In this model we can count the number of connected (congruent) dynamical units $\sharp\{\mathbf{1}_E\}$ and $\sharp\{\mathbf{1}_\mathbf{p}\}$. In this way for a given action of motion w the characterization of energy E_w and of momentum $\mathbf{p}_w$ becomes measurable. By means of the physical model $w_\mathbf{1} * \ldots * w_\mathbf{1}$ action of motion w is metricized.

3.1.1 Dynamical Unit

In everyday life we deal with various sources of energy. For example a wristwatch is driven by a battery or simply a spring. When the spring $\mathcal{S}$ is charged and locked it becomes a potential source of energy. The loaded spring - symbolized by $\mathcal{S}_E$ - has the potential E to cause an action. This energetic unit can be charged, reoriented in space and discharged in a reversible way (see figure 5).

So far our examples introduce possible causes for actions of motion (German: $\mathcal{W}$irkung) w: the action of a $\mathcal{B}$ow $w_\mathcal{B}$ and the action of a $\mathcal{C}$rossbow $w_\mathcal{C}$ in figure 4 and the action of a

$\mathcal{S}$pring $w_{\mathcal{S}}$ in figure 5. Despite the differences in the inner course of events - those actions have in common that in the initial state a resting cause (German: $\mathcal{U}$rsache) $\mathcal{U}_E\big|_{\mathbf{v}=\mathbf{0}}$ with potential of action E is coupled into a system of equally resting objects $\textcircled{a}_{\mathbf{v}=\mathbf{0}}$. When the cause $\mathcal{U}_E$ is unlocked (for example if $\mathcal{B}$ob intervenes - as a controlling physicist - and removes an external lock) it expends the potential of action E against objects $\textcircled{a}$ and $\textcircled{b}$. The latter act - under the effect of the external cause according to the Principle of Inertia [2] - against changes in their state of motion. In the final state the objects $\textcircled{a}$ and $\textcircled{b}$ fly apart into diametrically opposed directions and pick up against one another directed impulses {2.3}.[10] The discharged source of energy $\mathcal{U}\big|_{\mathbf{0}}$ stays behind as a resting object - now without potential of action E. The cause $\mathcal{U}$ acts against objets $\textcircled{a}$ and $\textcircled{b}$. (Despite differences in their inner progression and magnitude) in these simple actions the release of energy by the external cause $\mathcal{U}$ and the generation of momentum on objects $\textcircled{a}$ and $\textcircled{b}$ are inseparably unified.

Remark 7 *Energy and Momentum are distinguishable aspects but inseparably unified in an interaction of motion.*

We introduce a measurement method where a sequence of congruent actions is set up and coupled. The course of their couplings is controlled from the outside by (a collective of) physicists (see figure 7). Therefore in every individual action it only matters which change in the state of motion is ultimately attained - irrespective of details in its spatiotemporal progression. For the matching concatenation of a sequence of actions the physicist controls the initiation of each individual action - timely and at suitable position - such that the desired effect is achieved.

To conduct a physical measurement $\mathcal{A}$lice may pick out a *standard* from the variety of possible actions of motions which is reproducible and available anywhere and anytime and in any number. For example $\mathcal{A}$lice may provide a reservoir with springs $\mathcal{S}$ and with objects $\textcircled{a}$ which have the same dynamical behavior. Each two objects $\textcircled{a} \sim_{m^{(\mathrm{inert})}} \textcircled{b}$ have the same inertia when their impulse behavior is compared according to equivalence relation in Definition 3. With respect to inertia both objects are of the same kind. Their inertia is identical $m^{(\mathrm{inert})}_{\textcircled{a}} = m^{(\mathrm{inert})}_{\textcircled{b}}$. We will call them unit objets $\textcircled{1}$ from now on.

Similarly each two springs $\mathcal{S} \sim_E \tilde{\mathcal{S}}$ are set up with same energy. For each two actions $w:\ \mathcal{U}_E\big|_{\mathbf{0}}, \textcircled{a}_{\mathbf{0}}, \textcircled{a}_{\mathbf{0}} \ \Rightarrow\ \mathcal{U}\big|_{\mathbf{0}}, \textcircled{a}_{\mathbf{v}}, \textcircled{a}_{-\mathbf{v}}$ and $\tilde{w}:\ \tilde{\mathcal{U}}_{\tilde{E}}\big|_{\mathbf{0}}, \textcircled{a}_{\mathbf{0}}, \textcircled{a}_{\mathbf{0}} \ \Rightarrow\ \tilde{\mathcal{U}}\big|_{\mathbf{0}}, \textcircled{a}_{\tilde{\mathbf{v}}}, \textcircled{a}_{-\tilde{\mathbf{v}}}$ $\mathcal{A}$lice can check that both causes of action $\mathcal{U} \sim_E \tilde{\mathcal{U}}$ have the same energy by comparing their potential effect according to equivalence relation in Definition 4. With respect to final state of motion both energetic causes $\mathcal{U}_E$ and $\tilde{\mathcal{U}}_{\tilde{E}}$ have expended the same potential of action E. According to the Equipollence Principle their energy is identical $E_{\mathcal{U}} = \tilde{E}_{\tilde{\mathcal{U}}}$. We will call $\mathcal{A}$lice standard causes of action from now on energetic unit $\mathbf{1}_E$.

Definition 5 *For the purpose of basic dynamical measurements the (collective of) physicists provide a reservoir* $\{\mathbf{1}_E, \textcircled{1}\}$ *with dynamical measurement devices:*

[10]The attained *potential* momentum of the moving object $\textcircled{a}_{\mathbf{v}}$ is *realized* - likewise according to the Principle of Inertia - as impulse, impact, impetus against an external obstacle under the *condition* that the latter has been placed into the way of the moving object $\textcircled{a}_{\mathbf{v}}$.

- *standardized objects with equivalent impulse behavior - named unit object and symbolized by* ①

- *standardized causes* $\mathbf{1}$ *with equivalent potential* E *to cause a unit action* $w_{\mathbf{1}}$[11] *- named energetic unit and symbolized by* $\mathbf{1}_E$

$$w_{\mathbf{1}}: \quad \mathbf{1}_E\big|_{\mathbf{0}}\,,\ ①_{\mathbf{0}}\,,\ ①_{\mathbf{0}} \quad \Rightarrow \quad ①_{\mathbf{v_1}}\,,\ ①_{-\mathbf{v_1}} \tag{17}$$

- *with unit objects* ① *boosted into diametrically opposed direction with unit velocity* $\mathbf{v_1}$

- *two diametrically opposed unit momenta* $①_{\mathbf{v_1}}$ *and* $①_{-\mathbf{v_1}}$ *reproduce (if discharged source, e.g. neutral spring set up and unlocked at right place and time) a reversible unit action*

$$w_{\mathbf{1}}^{-1}: \quad ①_{\mathbf{v_1}}\,,\ ①_{-\mathbf{v_1}} \quad \Rightarrow \quad \mathbf{1}_E\big|_{\mathbf{0}}\,,\ ①_{\mathbf{0}}\,,\ ①_{\mathbf{0}} \quad .$$

D'Alembert utilizes in his *Traité de dynamique* congruent actions of a spring. He discusses the compression of equivalent springs up to a fixed mark. This is a very instructive approach Schlaudt remarks [23]. The action is quantified - not by the depth of compression (in one spring) but instead - by the number of springs which are compressed by a fixed distance. In this way one *abstracts* completely from the *inner dynamics* of the compression process.

All unit actions $w_{\mathbf{1}}$ - set up by $\mathcal{A}$lice in her reservoir $\{\mathbf{1}_E\big|_{\mathbf{0}}, ①_{\mathbf{0}}\}$ - are congruent to one another because all energetic causes $\mathbf{1}_E\big|_{\mathbf{0}}$ are initially at rest and because they all act against initially resting objects $①_{\mathbf{0}}$ in a *physically equivalent* way. The quantification of basic dynamical measures will boil down to counting those congruent dynamical units in an organized way.

3.1.2 Concatenation

We will construct dynamical models by means of reproducible configurations of reversible unit actions $w_{\mathbf{1}}$.

Definition 6 *Two simple actions of motion*

$$\begin{aligned} w: &\quad \mathcal{U}_E\big|_{\mathbf{v}_{\mathcal{U}}}\,,\ ⓐ_{\mathbf{v}_a}\,,\ ⓑ_{\mathbf{v}_b} \ \Rightarrow\ ⓐ_{\mathbf{v}'_a}\,,\ ⓑ_{\mathbf{v}'_b} \\ \tilde{w}: &\quad \tilde{\mathcal{U}}_{\tilde{E}}\big|_{\mathbf{v}_{\tilde{\mathcal{U}}}}\,,\ ⓑ_{\tilde{\mathbf{v}}_b}\,,\ ⓒ_{\tilde{\mathbf{v}}_c} \ \Rightarrow\ ⓑ_{\tilde{\mathbf{v}}'_b}\,,\ ⓒ_{\tilde{\mathbf{v}}'_c} \end{aligned}$$

are associated - in a diachronic way - at the same object $ⓑ_{\mathbf{v}'_b} \equiv ⓑ_{\tilde{\mathbf{v}}_b}$ *in the same state of motion* $\mathbf{v}'_b \stackrel{!}{=} \tilde{\mathbf{v}}_b$.

[11] In the complete formulation of the unit action

$$w_{\mathbf{1}}: \quad \mathbf{1}_E\big|_{\mathbf{0}}\,,\ ①_{\mathbf{0}}\,,\ ①_{\mathbf{0}} \quad \Rightarrow \quad \mathbf{1}\big|_{\mathbf{0}}\,,\ ①_{\mathbf{v_1}}\,,\ ①_{-\mathbf{v_1}}$$

the presence of the discharged energetic cause $\mathbf{1}$ (i.e. the discharged source after expending energy E) is suppressed in the final state. For simplicity we will only use the simplified notation.

In figure 6a we concatenate two unit actions $w_{\mathbf{1}} * w_{\mathbf{1}}^{-1}$ at unit object $①_{\mathbf{v}=0}$ which is in the intermediate state at rest. In figure 7 we concatenate a sequence of (transversal) actions $w_T^{(i)} * w_T^{(j)} * w_T^{(k)} * \ldots$ at same object $①_{v_{(1)}}$ which in each intermediate state moves freely with same velocity $v_{(1)}$.

Remark 8 *For inertial observer $\mathcal{A}$lice a generic action of motion is caused by a moving source of energy $\mathcal{U}_E\big|_{\mathbf{v}_{\mathcal{U}}}$ against initially moving objects $ⓐ_{\mathbf{v}_a}$ and $ⓑ_{\mathbf{v}_b}$*

$$w: \ \mathcal{U}_E\big|_{\mathbf{v}_{\mathcal{U}}}, \ ⓐ_{\mathbf{v}_a}, \ ⓑ_{\mathbf{v}_b} \ \Rightarrow \ ⓐ_{\mathbf{v}'_a}, \ ⓑ_{\mathbf{v}'_b}$$

A (uniformly) moving observer $\mathcal{B}$ob will see this action involving the same physical objects $\mathcal{U}_E\big|_{\mathbf{v}_{\mathcal{U}}}$, $ⓐ_{\mathbf{v}_a}$ and $ⓑ_{\mathbf{v}_b}$. $\mathcal{A}$lice and $\mathcal{B}$ob specify initial and final state of motion of same objects $i = \mathcal{U}, a, b, a', b'$ in a <u>covariant</u> way [24]

$$\mathbf{v}_i \ = \ \mathbf{v}_i^{(\mathcal{A})} \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{A})}} \ = \ \mathbf{v}_i^{(\mathcal{B})} \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{B})}} \ . \tag{18}$$

When $\mathcal{A}$lice is uniformly moving relative to $\mathcal{B}$ob $\mathbf{v}_{\mathcal{A}} = \mathbf{v}_{\mathcal{A}}^{(\mathcal{B})} \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}$ then their measured values (German: Meßwerte) of motion for same objects $\mathcal{U}_E$, $ⓐ_{v_a}$ and $ⓑ_{v_b}$ transform - under the condition of Galilei Kinematics - simply by vectorial addition

$$\mathbf{v}_i^{(\mathcal{B})} \ = \ \mathbf{v}_i^{(\mathcal{A})} \ + \ \mathbf{v}_{\mathcal{A}}^{(\mathcal{B})} \tag{19}$$

for each object $i = \mathcal{U}, a, b, a', b'$.

For example $\mathcal{A}$lice unit action $w_{\mathbf{1}}^{(\mathcal{A})}$ set up with resting energetic unit $\mathbf{1}_E\big|_0$ and unit objects $①_0$ from her Reservoir

$$w_{\mathbf{1}}^{(\mathcal{A})}: \ \ \mathbf{1}_E\big|_{\mathbf{0}}, \ ①_{\mathbf{0}}, \ ①_{\mathbf{0}} \ \Rightarrow \ ①_{\mathbf{v}_1}, \ ①_{-\mathbf{v}_1}$$

will be seen by Bob (relative to whom $\mathcal{A}$lice moves with uniform velocity $\mathbf{v}_{\mathcal{A}}$) as a simple action of motion of the same physical objects

$$w_{\mathbf{1}}^{(\mathcal{A})}: \ \ \mathbf{1}_E\big|_{\mathbf{v}_{\mathcal{A}}}, \ ①_{\mathbf{v}_{\mathcal{A}}}, \ ①_{\mathbf{v}_{\mathcal{A}}} \ \Rightarrow \ ①_{\mathbf{v}_1+\mathbf{v}_{\mathcal{A}}}, \ ①_{-\mathbf{v}_1+\mathbf{v}_{\mathcal{A}}}$$

where both energetic source $\mathbf{1}_E\big|_{\mathbf{v}_{\mathcal{A}}}$ and objects $①_{\mathbf{v}_{\mathcal{A}}}$, $①_{\mathbf{v}_{\mathcal{A}}}$ initially move with velocity $\mathbf{v}_{\mathcal{A}}$ (see figure 6b).

The unit action $w_{\mathbf{1}}^{(\mathcal{A})}$ $\mathcal{A}$lice sets up from her reservoir is a reversible inelastic collision. For a possible realization of such unit action by utilization of a spring see figure 5. The *elastic concatenation* of $\mathcal{A}$lice unit actions $w_{\mathbf{1}} * w_{\mathbf{1}}^{-1}$ will be seen by suitably moving observer $\mathcal{B}$ob as an 'elastic transversal collision' (see figure 6c).

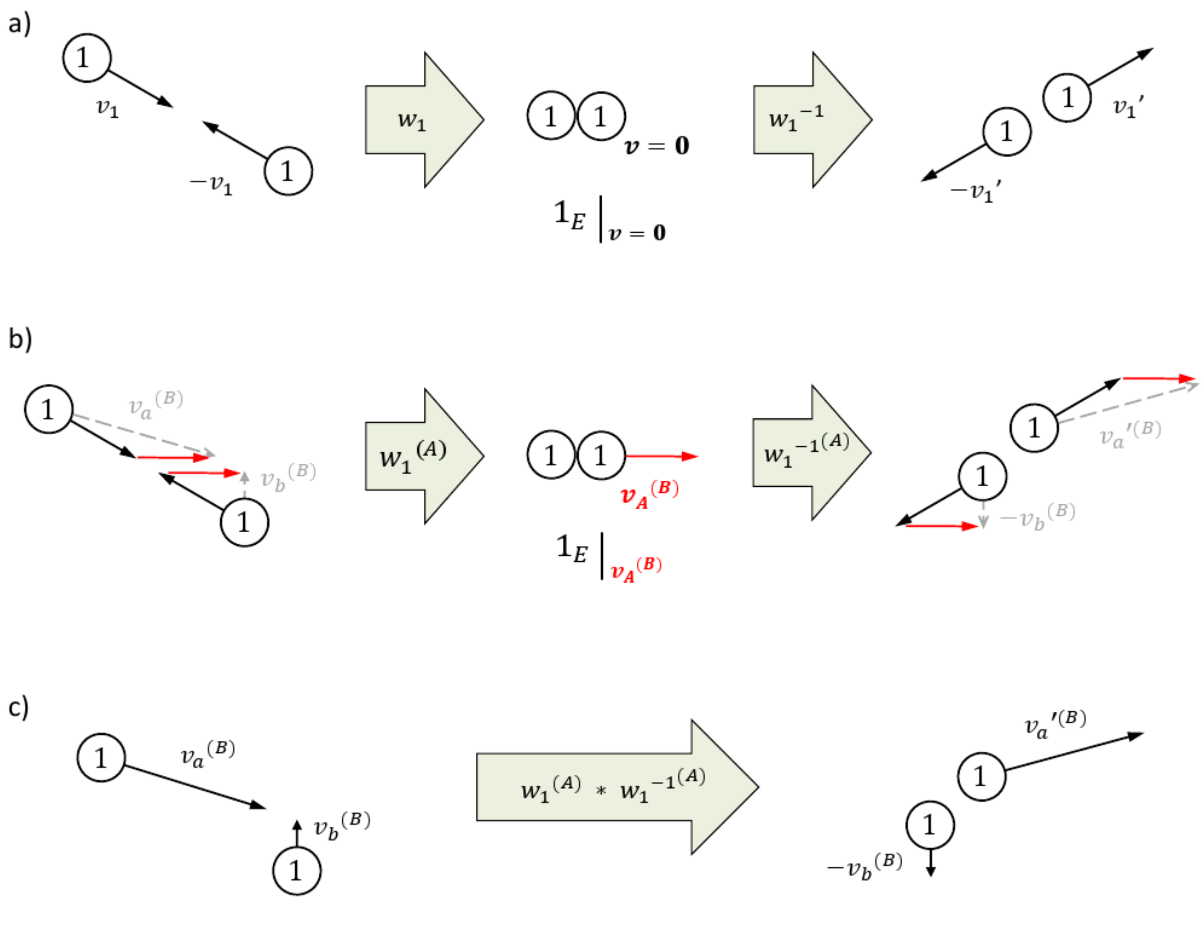

Figure 6: a) reversible unit action $w_{\mathbf{1}}$ and elastic composition $w_{\mathbf{1}} * w_{\mathbf{1}}^{-1}$ for $\mathcal{A}$lice b) covariant transformation to perspective of $\mathcal{B}$ob c) composite action: 'elastic transversal collision'

3.1.3 Physical Model

We physically specify generic interaction of motion w in a system of material objects $\{G_i\}$. The state of motion $\mathbf{v}_i$ of involved objects G_i changes throughout its spatiotemporal progression. We construct a replacement process which reproduces the kinetic effect of action w on each individual element G_i with regard to gained changes in state of motion $\mathbf{v}_i$.

We quantify intrinsic action w by means of unit actions $w_{\mathbf{1}}$. We presuppose both actions solely as completed processes with regard to changes in their final state of motion. We do not presuppose any mathematical relation regarding the course of their inner progression. Thus unit actions of motion $w_{\mathbf{1}}$ taken by themselves are also *unquantified* - but these units are *congruent* among one another. By coupling dynamical units from an external reservoir $\left\{\mathbf{1}_E\big|_{\mathbf{0}}, \textcircled{1}_{\mathbf{0}}\right\}$ we concatenate a sequence of unit actions $w_{\mathbf{1}} * \ldots * w_{\mathbf{1}}$. The course of their couplings is controlled from outside by (a collective of) physicists. They assure that each unit action is initiated at right place and moment and in the right way such that the desired effect is achieved (see figure 7). This physical model reproduces the kinetic effect of intrinsic action w

$$w \quad \sim_{E,\mathbf{p}} \quad w_{\mathbf{1}} * \ldots * w_{\mathbf{1}}$$

with regard to energy and momentum characterization.

The layout of congruent dynamical units $w_{\mathbf{1}}$ is designed so that it reproduces - separately for each individual object G_i of system $\{G_i\}$ - the resulting changes in the state of motion of action w (irrespective of details in its spatiotemporal progression). In this replacement process the physicist can count the number of congruent dynamical units $\sharp\{\mathbf{1}_E\}$ and $\sharp\{\mathbf{1}_{\mathbf{p}}\}$. By means of such physical models we specify interactions of motions in an intersubjectively reproducible way. The intrinsically well-defined, pre-theoretical ordering relations with regard to energy and momentum {2.3} become measurable. By means of physical model $w_{\mathbf{1}} * \ldots * w_{\mathbf{1}}$ the energy E_w and momentum $\mathbf{p}_w$ of generic interaction of motion w is metricized. The genetic interrelation of basic dynamical measures (postulated as equations of motion in axiomatic systems) will be justified measurement theoretically - without mathematical presuppositions.

3.1.4 Quantification of Elastic Collision

Theorem 1 *Consider a reservoir with equivalent physical objects* $\{\textcircled{1}\}$. *Suppose we can tightly connect* n *of them* $\underbrace{\textcircled{1} * \ldots * \textcircled{1}}_{n\times} =: \textcircled{n}$ *such that the composite acts like a single rigid body (see figure 1). In an elastic collision* w *between different composites of equivalent objects* $\textcircled{1} * \ldots * \textcircled{1}$ *and* $\textcircled{1}$ *their respective velocities change - in Galilei Kinematics - according to*

$$w: \quad \textcircled{n}_{1\cdot\mathbf{v}}\,,\ \textcircled{1}_{-n\cdot\mathbf{v}} \quad \Rightarrow \quad \textcircled{n}_{-1\cdot\mathbf{v}}\,,\ \textcircled{1}_{+n\cdot\mathbf{v}} \quad . \tag{20}$$

We examine the elastic collision between two generic objects as in {1}. Without restricting generality we assume both objects are (rigid) composites of unit objects $\textcircled{1}$. We do not presuppose the way in which velocities of two generic objects change in an elastic collision.[12] The *trick* is to mediate their direct collision by an indirect replacement process. The physical model solely consists of elastic collisions between equivalent objects ① which must behave in a symmetrical way. The construction is based on physical and kinematical principles:

- *Existence* of symmetric elastic collisions between equivalent objects - as realized e.g. in the elastic concatenation of reversible unit actions $w_{\mathbf{1}} * w_{\mathbf{1}}{}^{-1}$ (see figure 6c).
- For all uniformly moving observers the *kinematical description* of respective states of motion in the same collision transforms Galilei covariant (see Remark 8).
- Possibility to concatenate configurations of *intrinsically congruent* dynamical units.
- *Impossibility of a Perpetuum Mobile*: If coupled into a circular process both (reversible) processes are equivalent with regard to energy and momentum.

Therefore our replacement process is universal. After direct elastic collision and after indirect replacement process the two composite objects gain same final state of motion.

In the initial state we have two incident objects: unit object $①_{v_{(n)}}$ with velocity $v_{(n)}$ moves into opposite direction of composite object $ⓝ_{v_{(1)}}$ with velocity $v_{(1)}$. When we unlock its inner binding the composite $① * \ldots * ①$ becomes a swarm of n unit objects $①_{v_{(1)}}, \ldots, ①_{v_{(1)}}$ with initial velocity $v_{(1)}$. We couple dynamical units from an external reservoir separately against all unit elements ① of both incident objects. Each gets reversed in a separate replacement process. (A collective of) physicists set up and control the process in a series of congruent transversal kicks w_T. Their construction method involves 3 steps:

1. They independently *fire* dynamical units from an external reservoir $\left\{\mathcal{U}_\epsilon\big|_{\mathbf{0}}, ①_{\mathbf{0}}\right\}$ against incident particles $①_{v_{(1)}}, \ldots, ①_{v_{(1)}}$ and $①_{v_{(n)}}$. They *generate* elastic transversal collisions w_T which successively turn around those particles kick after kick (see figure 10).
2. Separately against each particle they *control a sequence* of congruent transversal kicks $w_T * \ldots * w_T$ to reverse its direction of motion (see figure 7).
3. They align the impulse reversion processes for all incident particles such that - pairwise at diametrically opposite locations - all dynamical units can be *recycled* back into the external reservoir $\left\{\mathcal{U}_\epsilon\big|_{\mathbf{0}}, ①_{\mathbf{0}}\right\}$ (see figure 8).

In the end the impulse of each incident object $\{①_{-v_{(1)}}, \ldots, ①_{-v_{(1)}}\} \equiv ⓝ_{-v_{(1)}}$ and $①_{-v_{(n)}}$ is exactly reversed. Physicists mediate their 'indirect collision' by means of dynamical units

[12] An impartial approach to basic dynamical measures is not built upon anticipation of equations for conserved quantities nor on the assumption of mathematical relations which basic measures eventually satisfy. We do not yet have basic dynamical measures nor their mathematical formulation. Their magnitude is ordered in a physical comparison. Each measured value follows directly from basic measurements. We will derive relations between different dynamical measures from the interrelation of physical conditions in their respective underlying basic measurements.

from an external reservoir. Those were temporarily expended but finally all recycled back into the reservoir. Every act of the physicists is reversible. Our physical model reproduces the direct collision of two composite objects with regard to energy and momentum. The replacement process is equivalent to their direct elastic collision. We illustrate the construction method for a simple configuration.

Proposition 1 *Consider an elastic collision of three equivalent objects: One unit object* $\textcircled{1}_{v_{(2)}}$ *comes in from left with initial velocity* $v_{(2)}$ *and two unit objects* $\textcircled{1}_{R_{15^\circ}v_{(1)}}$ *and* $\textcircled{1}_{R_{-15^\circ}v_{(1)}}$ *come in from right with velocity* $v_{(1)}$ *under orientation* 15° *resp.* -15° *(see figure 8). In the final state the motion of all objects is exactly reversed*

$$w: \ \textcircled{1}_{v_{(2)}}\, , \ \textcircled{1}_{R_{15^\circ}v_{(1)}}\, , \ \textcircled{1}_{R_{-15^\circ}v_{(1)}} \ \Rightarrow \ \textcircled{1}_{-v_{(2)}}\, , \ \textcircled{1}_{-R_{15^\circ}v_{(1)}}\, , \ \textcircled{1}_{-R_{-15^\circ}v_{(1)}} \ . \tag{21}$$

Proof: In **step I** $\mathcal{A}$lice and $\mathcal{B}$ob prepare elastic transversal collisions. They have access to a reservoir of equivalent dynamical units $\left\{\mathcal{U}_\epsilon\big|_{\mathbf{0}}, \textcircled{1}_{\mathbf{0}}\right\}$. They - temporarily - expend energy units $\mathcal{U}_\epsilon\big|_{\mathbf{0}}$ against resting unit objects $\textcircled{1}_{\mathbf{0}}$ from the reservoir

$$w_\epsilon: \ \mathcal{U}_\epsilon\big|_{\mathbf{0}}\, , \ \textcircled{1}_{\mathbf{0}}\, , \ \textcircled{1}_{\mathbf{0}} \ \Rightarrow \ \textcircled{1}_{\epsilon\cdot\mathbf{v_1}}\, , \ \textcircled{1}_{-\epsilon\cdot\mathbf{v_1}}$$

to *prepare* (congruent) transversal impulses $\textcircled{1}_{\epsilon\cdot\mathbf{v_1}}$ (see figure 9a). $\mathcal{A}$lice and $\mathcal{B}$ob can set up the angle θ for their congruent unit actions

$$w_\epsilon^{(\theta)}: \ \mathcal{U}_\epsilon\big|_{\mathbf{0}}\, , \ \textcircled{1}_{\mathbf{0}}\, , \ \textcircled{1}_{\mathbf{0}} \ \Rightarrow \ \textcircled{1}_{\mathrm{R}_\theta(\epsilon\cdot\mathbf{v_1})}\, , \ \textcircled{1}_{\mathrm{R}_\theta(-\epsilon\cdot\mathbf{v_1})} \tag{22}$$

and fire their reservoir particles with velocity $\epsilon \cdot v_{\mathbf{1}}$ into any suitable direction θ. They pick initially resting object $\textcircled{1}$ from the external reservoir $\{\textcircled{1}_{\mathbf{0}}\}$ and catapult it into the way of incident object $\textcircled{1}_{v_{(1)}}$ resp. $\textcircled{1}_{v_{(2)}}$. Each transversal kick from a reservoir particle $\textcircled{1}_{\epsilon\cdot v_{\mathbf{1}}}$ is an elastic collision against incident objects $\textcircled{1}_{v_{(1)}}$ resp. $\textcircled{1}_{v_{(2)}}$ (see figure 9b)

$$w_T: \ \textcircled{1}_{v_{(i)}}\, , \ \textcircled{1}_{\epsilon\cdot\mathbf{v_1}} \ \Rightarrow \ \textcircled{1}_{v'_{(i)}}\, , \ \textcircled{1}_{-\epsilon\cdot\mathbf{v_1}} \ .$$

After each kick recoil particle $\textcircled{1}_{-\epsilon\cdot\mathbf{v_1}}$ bounces off with same velocity into opposite direction while incident particle moves on freely with same velocity $v'_{(i)} = \mathrm{R}_{\alpha_i}v_{(i)}$ into a direction which is rotated by angle α_i (depending on the velocity $v_{(i)}$ of incident object $\textcircled{1}_{v_{(1)}}$ resp. $\textcircled{1}_{v_{(2)}}$).[13] The elastic transversal collision $w_T = w^{(\mathcal{O})} * w^{-1(\mathcal{O})}$ is realizable as concatenation of two reversible simple actions $w^{(\mathcal{O})}$ of a suitably moving $\mathcal{O}$bserver (see figure 6c). $\mathcal{A}$lice and $\mathcal{B}$ob *provide* (congruent) elastic transversal collisions with various expedient orientations θ

$$w_T^{(\theta)} \ := \ w_\epsilon^{(\theta)} * \ w_T \ . \tag{23}$$

Each successively rotates direction of motion of incident object $\textcircled{1}_{v_{(i)}}$ by angle α_i for $i = 1, 2$.

[13] In an elastic transversal collision with reservoir particle $\textcircled{1}_{\epsilon\cdot v_{\mathbf{1}}}$ the incident object $\textcircled{1}_{v_{(i)}}$ is kicked around angle $\alpha_1 = 60^\circ$ resp. $\alpha_2 = 30^\circ$ if it has initial velocity $v_{(i)} = \sin^{-1}\left(\frac{\alpha_i}{2}\right) \cdot \epsilon \cdot v_{\mathbf{1}}$ for $i = 1, 2$ (see Lemma 1).

In **step II** $\mathcal{A}$lice controls the impulse reversion process for incident object $①_{v_{(1)}}$ from the right. She controls preparation and coupling of congruent actions (23) from her external reservoir. In figure 7 the coordinated effort of a team of assistants is required to set up and control a sequence of transversal collisions: Three women have to line up at the corners and obey instructions to timely operations in the right way in order to reverse the motion of incident particle $①_{v_{(1)}}$. Everybody has to know *when* and *where* to pick the next initially resting object ① from the reservoir and *how* to fire it into the way of incident object $①_{v_{(1)}}$. The course of their couplings is controlled from the outside by (a collective of) physicists. $\mathcal{A}$lice directs the initiation of each individual action - timely and at suitable position - such that the impulse reversion for object $①_{v_{(1)}}$ is achieved. In figure 7 $\mathcal{A}$lice team associates a sequence of three (transversal) actions

$$W_{(1)} \; := \; w_T^{(30°)} * w_T^{(90°)} * w_T^{(150°)} \tag{24}$$

at same object $①_{v_{(1)}}$ which in each intermediate state moves freely with same velocity $v_{(1)}$. Each (congruent) transversal impulse $①_{\epsilon \cdot v_{\mathbf{1}}}$ turns incident object $①_{v_{(1)}}$ through another 60°. After three successive kicks its direction of motion is reversed.

Similarly $\mathcal{B}$ob controls a separate impulse reversion process for incident object $①_{v_{(2)}}$. Since it comes in faster $v_{(2)} > v_{(1)}$ the effect of every congruent action (23) on the direction of motion is smaller. $\mathcal{B}$ob sets up and controls more elastic transversal collisions against unit object $①_{v_{(2)}}$ with velocity $v_{(2)}$: Six men line up at the corners and know how to fire the next reservoir object ① into the way of incident object $①_{v_{(2)}}$. $\mathcal{B}$ob directs the initiation of each individual kick - timely and at suitable position - such that the impulse of object $①_{v_{(2)}}$ gets reversed. In figure 7 $\mathcal{B}$ob's team associates a sequence of six (transversal) actions

$$W_{(2)} \; := \; w_T^{(15°)} * w_T^{(45°)} * \ldots * w_T^{(165°)} \tag{25}$$

at same object $①_{v_{(2)}}$ which in each intermediate state moves freely with same velocity $v_{(2)}$. Each (congruent) transversal impulse $①_{\epsilon \cdot v_{\mathbf{1}}}$ turns incident object $①_{v_{(2)}}$ through another 30°. After six successive kicks its direction of motion is reversed.

In **step III** $\mathcal{A}$lice and $\mathcal{B}$ob align their impulse reversion processes $W_{(1)}$ and $W_{(2)}$ such that they can recycle all their expended dynamical units back into the reservoir $\left\{\mathcal{U}_\epsilon\big|_{\mathbf{0}}, ①_{\mathbf{0}}\right\}$. $\mathcal{A}$lice aligns her two impulse reversion processes $W_{(1)}$ for the two incident objects $①_{v_{(1)}}, ①_{v_{(1)}}$ with the orientation of Bob's impulse reversion process $W_{(2)}$ for incident object $①_{v_{(2)}}$ such that all elastic transversal collisions are aligned pairwise antiparallel - along dashed lines in figure 8 - at diametrically opposite locations.

$\mathcal{A}$lice rotates the whole configuration of her impulse reversion process by angle $\beta = 195°$. After reorientation - symbolized by rotation operator $\mathrm{R}_\beta[\;\cdot\;]$ - the incident particle $①_{v_{(1)}}$ comes in with velocity $v_{(1)}$

$$\mathrm{R}_\beta\left[①_{v_{(1)}}\right] \; = \; ①_{R_\beta v_{(1)}}$$

under orientation $\beta = 195°$. The rotation operator R_β represents an instruction from $\mathcal{A}$lice to build up the physical model with a modified orientation. Her team of physicists prepares

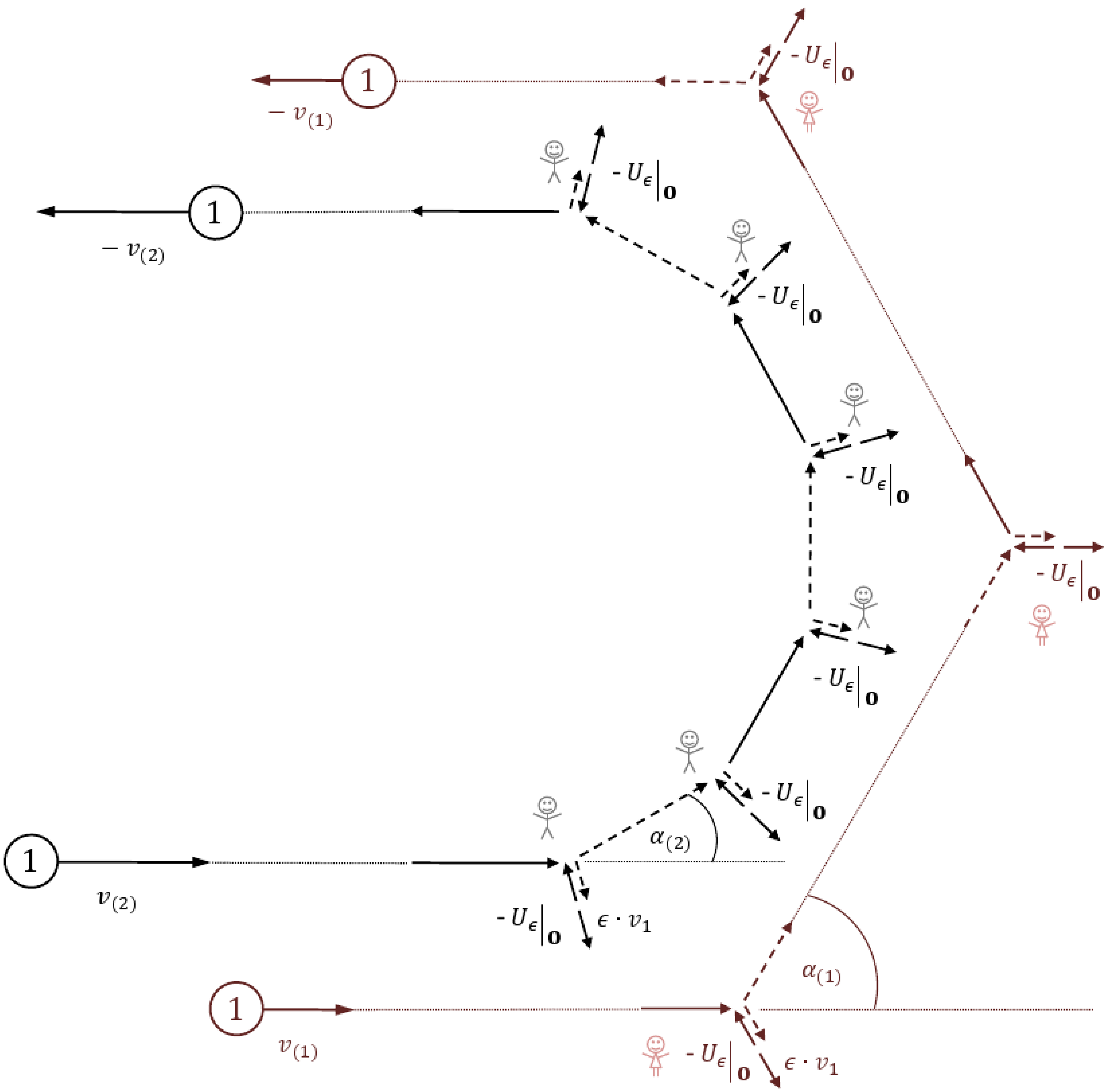

Figure 7: In a coordinated effort $\mathcal{A}$lice and $\mathcal{B}$ob's team of physicists set up and control a diachronic sequence of elastic transversal collisions w_T in order to provide impulse reversion process $W_{(1)}$ and - according to matching condition $\alpha_1 \stackrel{!}{=} 2 \cdot \alpha_2$ - analogous for $W_{(2)}$

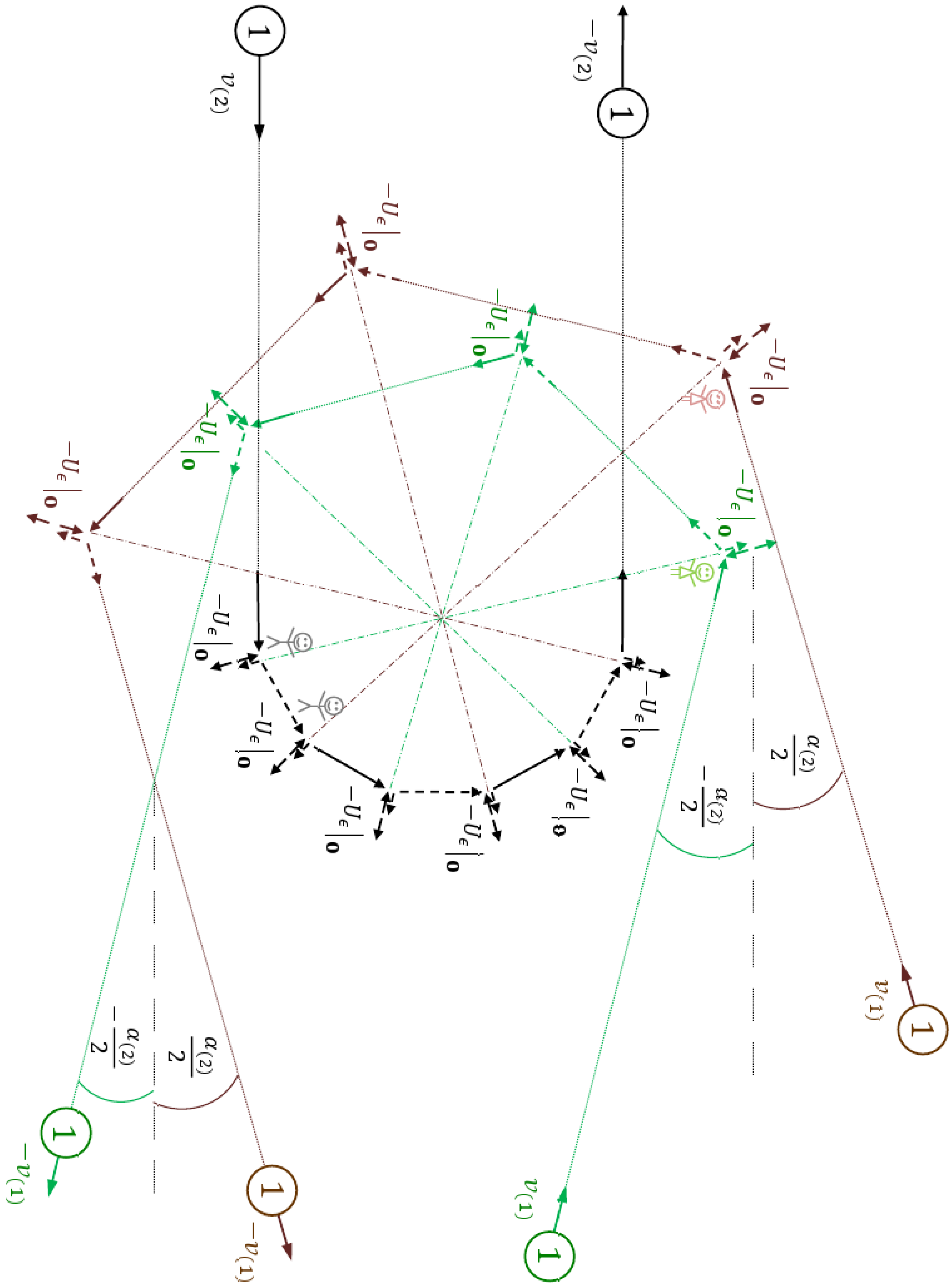

Figure 8: align $2\times$ impulse reversion process $W_{(1)}$ and $1\times$ impulse reversion process $W_{(2)}$

all transversal impulses (22) by expending congruent energy unit $\mathcal{U}_\epsilon\big|_{\mathbf{0}}$ into direction $\theta+\beta$

$$\mathrm{R}_\beta\left[w_\epsilon^{(\theta)}\right] \;=\; w_\epsilon^{(\theta+\beta)}$$

and they couple all elastic transversal collisions (23) rotated by additional angle $\beta = 195°$ (in the same plane). Finally her complete impulse reversion process for first incident particle $①_{v_{(1)}}$

$$\mathrm{R}_\beta\left[w_T^{(30°)} * w_T^{(90°)} * w_T^{(150°)}\right] \;=\; w_T^{(30°+\beta)} * w_T^{(90°+\beta)} * w_T^{(150°+\beta)}$$

is rotated by $\beta = 195°$. Similarly $\mathcal{A}$lice rotates the impulse inversion process $\mathrm{R}_{165°}\left[W_{(1)}\right]$ for the second incident particle $①_{v_{(1)}}$ by an angle $\beta = 165°$.

Every elastic transversal collision $w_T^{(\theta)} \;:=\; w_\epsilon^{(\theta)} * \; w_T$ picks two resting unit objects $①_{\mathbf{0}}$ from the reservoir and generates two recoil particles $①_{-\epsilon\cdot\mathbf{v_1}}$ with same velocity $-\epsilon\cdot\mathbf{v_1}$: one in the preparation $w_\epsilon^{(\theta)}$ (22) and the other after the elastic kick w_T (23). In order to recycle those recoil particles $\mathcal{A}$lice and $\mathcal{B}$ob associate their impulse inversion processes

$$W_{(2)} \;*\; \mathrm{R}_{165°}\left[W_{(1)}\right] \;*\; \mathrm{R}_{195°}\left[W_{(1)}\right] \tag{26}$$

pairwise in diametrically opposed recoil particles $①_{\epsilon\cdot v_\mathbf{1}}$ and $①_{-\epsilon\cdot v_\mathbf{1}}$ (along all dashed lines in figure 8). After inserting in (26) all their congruent elastic transversal collisions

$$\left\{w_T^{(15°)} * w_T^{(45°)} * \ldots * w_T^{(165°)}\right\} \;*\; \left\{w_T^{(30°+165°)} * w_T^{(90°+165°)} * w_T^{(150°+165°)}\right\}$$
$$* \;\left\{w_T^{(30°+195°)} * w_T^{(90°+195°)} * w_T^{(150°+195°)}\right\}$$

they divide in pairs of antiparallel elastic transversal collisions

$$\left(w_T^{(15°)} * w_T^{(30°+165°)}\right) * \left(w_T^{(45°)} * w_T^{(30°+195°)}\right) * \ldots * \left(w_T^{(165°)} * w_T^{(150°+195°)}\right) \quad .$$

Each associated tuple of four antiparallel recoil particles $①_{\epsilon\cdot\mathbf{v_1}}$, $①_{\epsilon\cdot\mathbf{v_1}}$, $①_{-\epsilon\cdot\mathbf{v_1}}$ and $①_{-\epsilon\cdot\mathbf{v_1}}$ reproduces the two - temporarily expended - congruent units of energy $\mathcal{U}_\epsilon\big|_{\mathbf{0}}$ and returns four resting particles back into the reservoir $\{①_{\mathbf{0}}\}$ (see figure 9c).

The association of $\mathcal{A}$lice and $\mathcal{B}$ob's impulse reversion processes (26) involves three incident objects: One unit object $①_{v_{(2)}}$ comes in from left with initial velocity $v_{(2)}$ and two unit objects $①_{R_{15°}v_{(1)}}$ and $①_{R_{-15°}v_{(1)}}$ come in from right with velocity $v_{(1)}$ under orientation $15°$ resp. $-15°$ (see figure 8). $\mathcal{A}$lice and $\mathcal{B}$ob mediate the impulse reversion of incident particles $①_{R_{15°}v_{(1)}}$, $①_{R_{-15°}v_{(1)}}$, $①_{v_{(2)}}$ by a sequence of congruent unit actions in 3 steps. For each transversal kick they temporarily expend energetic units which finally are all recycled back into the reservoir $\left\{\mathcal{U}_\epsilon\big|_{\mathbf{0}}, ①_{\mathbf{0}}\right\}$. After replacement process (26) only the impulse of incident objects $①_{-v_{(2)}}$, $①_{-R_{15°}v_{(1)}}$ and $①_{-R_{-15°}v_{(1)}}$ has changed. Their state of motion is exactly reversed. Every act in $\mathcal{A}$lice and $\mathcal{B}$ob's procedure is reversible. Both reversible processes - direct elastic collision

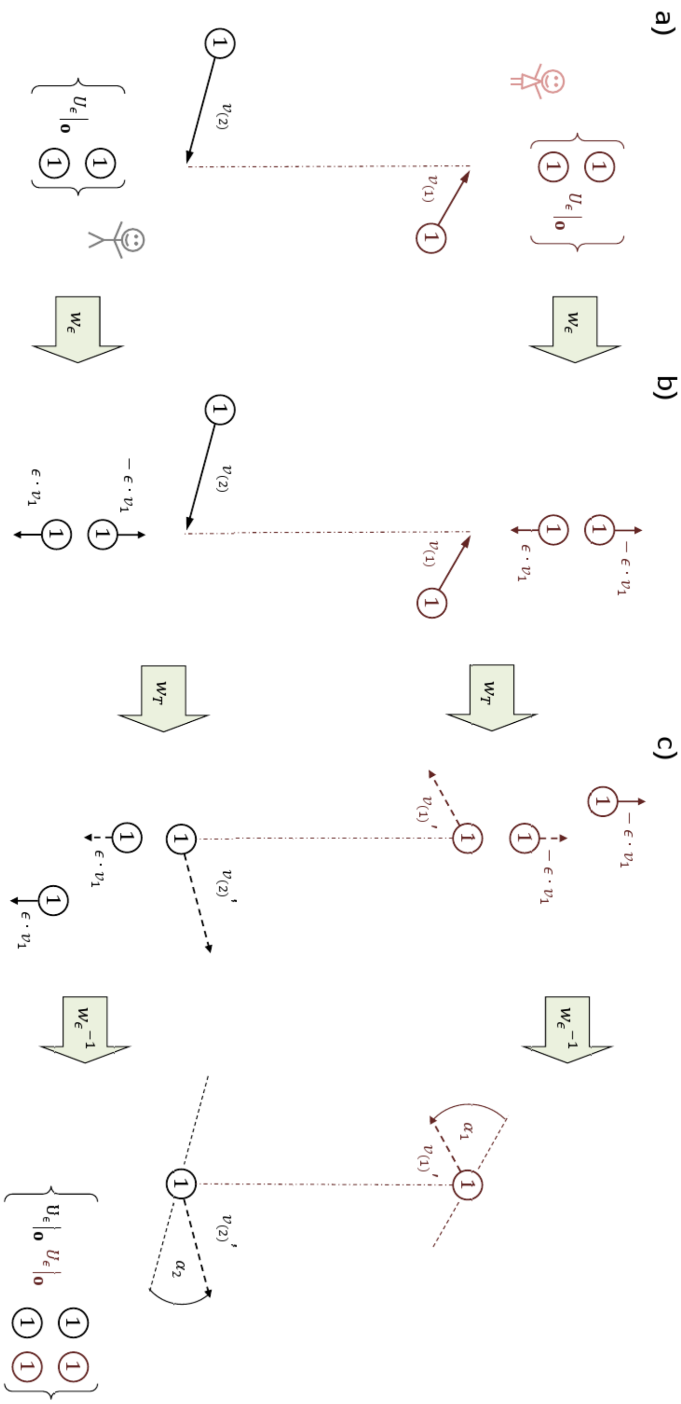

Figure 9: $\mathcal{A}$lice and $\mathcal{B}$ob at diametrically opposed positions set up and control process of a) expend energy unit $\mathcal{U}_\epsilon\big|_{\mathbf{0}}$ against $\textcircled{1}_{\mathbf{0}}$ and $\textcircled{1}_{\mathbf{0}}$ from their reservoir to provide two antiparallel transversal momenta for an b) elastic transversal collision with incident $\textcircled{1}_{\overrightarrow{v_{(1)}}}$ resp. $\textcircled{1}_{\overrightarrow{v_{(2)}}}$ c) antiparallel recoil particles reproduce the two - temporarily expended - units of energy $\mathcal{U}_\epsilon\big|_{\mathbf{0}}$ and return as resting particles back into the reservoir $\{\textcircled{1}_{\mathbf{0}}\}$

of three unit objects (21) and replacement process (26) set up by $\mathcal{A}$lice and $\mathcal{B}$ob - can be coupled into a circular process

$$W_{(2)} \ * \ \mathrm{R}_{165^\circ}\left[W_{(1)}\right] \ * \ \mathrm{R}_{195^\circ}\left[W_{(1)}\right] \quad * \quad w^{-1} \ .$$

By the impossibility of a Perpetuum Mobile both processes cause same changes in the final state of motion. The indirect impulse reversion process

$$W_{(2)} \ * \ \mathrm{R}_{165^\circ}\left[W_{(1)}\right] \ * \ \mathrm{R}_{195^\circ}\left[W_{(1)}\right] \quad \sim_{E,\mathbf{p}} \quad w$$

reproduces the direct elastic collision w (21) with regard to energy and momentum.

□

The method for the construction of replacement process (26) is to mediate the direct elastic collision - of our three incident objects $①_{v_{(1)}}$, $①_{v_{(1)}}$, $①_{v_{(2)}}$ - by means of congruent unit actions $w_\epsilon^{(\theta)}$ with an external reservoir $\left\{\mathcal{U}_\epsilon\big|_{\mathbf{0}}, ①_{\mathbf{0}}\right\}$. The recycling of those dynamical units - along dashed lines in figure 8 - works out because we have chosen suitable velocities $v_{(i)}$ and collision angles α_i for incident objects $①_{v_{(1)}}$, $①_{v_{(1)}}$, $①_{v_{(2)}}$. We have chosen velocity $v_{(i)} = \sin^{-1}\left(\frac{\alpha_i}{2}\right)\cdot\epsilon\cdot v_{\mathbf{1}}$ for incident object $i = 1, 2$ in accordance with the impetus of all reservoir particles $①_{\epsilon\cdot v_{\mathbf{1}}}$. Then the corresponding collision angles $\alpha_1 = 60^\circ$ resp. $\alpha_2 = 30^\circ$ allow for the construction of diametrically matching configuration (26). The kinematic characterization of a generic elastic transversal collision between two equivalent objects is given below.

Lemma 1 *In each elastic transversal collision between equivalent objects (see figure 10b)*

$$w_T : \quad ①_{v_{(i)}} \ , \ ①_{\epsilon\cdot v_{\mathbf{1}}} \ \Rightarrow \ ①_{v'_{(i)}} \ , \ ①_{-\epsilon\cdot v_{\mathbf{1}}}$$

$\mathcal{R}$eservoir particle $①_{\epsilon\cdot v_{\mathbf{1}}}$ kicks in from below with fixed velocity $\epsilon\cdot v_{\mathbf{1}}$ and rebounds antiparallel. Incident object $①_{v_{(i)}}$ moves on with same velocity $v'_{(i)} = R_{\alpha_i} v_{(i)}$ into a direction which is rotated by angle α_i satisfying

$$\sin\left(\frac{\alpha_i}{2}\right) = \frac{\epsilon}{v_{(i)}} \quad . \tag{27}$$

Proof: *Existence:* $\mathcal{A}$lice and $\mathcal{B}$ob observe the same elastic collision between two unit objects. $\mathcal{A}$lice can prepare the initial velocity of both objects

$$\mathbf{v}_{(i)} = \begin{pmatrix} h_{(i)} \\ -\epsilon \end{pmatrix} \cdot v_{\mathbf{1}^{(\mathcal{A})}} \qquad \mathbf{v}_{\mathcal{R}} = -\begin{pmatrix} h_{(i)} \\ -\epsilon \end{pmatrix} \cdot v_{\mathbf{1}^{(\mathcal{A})}}$$

with fixed horizontal and vertical components (see figure 10a).[14] A process which is an elastic collision for $\mathcal{A}$lice is an elastic collision for $\mathcal{B}$ob as well {3.1.1}.

[14] $\mathcal{A}$lice can adjust the scattering angle of an elastic collision $\tan(\frac{\alpha_i}{2}) = \frac{\epsilon}{h_{(i)}}$ freely. For example she can set up the rotation angle in an elastic association of two unit actions (see figure 6) or alternatively the impact parameter in an (controlled) elastic collision of two rigid balls.

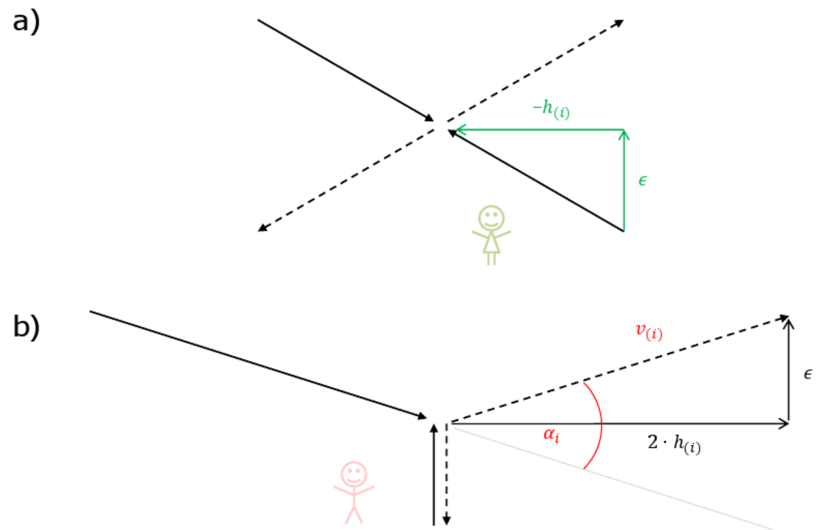

Figure 10: a) $\mathcal{A}$lice sets up scattering angle for a symmetric elastic collision b) same process appears as elastic transversal collision w_T for suitably moving observer $\mathcal{B}$ob

Kinematical characterization: Let $\mathcal{A}$lice move relative to $\mathcal{B}$ob with constant velocity

$$\mathbf{v}_{\mathcal{A}} = \begin{pmatrix} h_{(i)} \\ 0 \end{pmatrix} \cdot v_{\mathbf{1}^{(\mathcal{B})}}$$

in the horizontal direction. Measured values of motion for both colliding objects transform - Galilei covariant - by vectorial addition (see Remark 8). For $\mathcal{B}$ob incident object $①_{\mathbf{v}_{(i)}}$ has twice the horizontal velocity $2 \cdot h_{(i)} \cdot v_{\mathbf{1}^{(\mathcal{B})}}$ and vertical component $\epsilon \cdot v_{\mathbf{1}^{(\mathcal{B})}}$ in the collision

$$\mathbf{v}_{(i)} = \begin{pmatrix} 2 \cdot h_{(i)} \\ \mp\epsilon \end{pmatrix} \cdot v_{\mathbf{1}^{(\mathcal{B})}} \qquad \mathbf{v}_{\mathcal{R}} = \begin{pmatrix} 0 \\ \pm\epsilon \end{pmatrix} \cdot v_{\mathbf{1}^{(\mathcal{B})}}$$

while the $\mathcal{R}$eservoir particle $①_{\epsilon \cdot \mathbf{v}_{\mathbf{1}}}$ moves up and down vertically with same velocity $\epsilon \cdot v_{\mathbf{1}^{(\mathcal{B})}}$. For moving $\mathcal{B}$ob the same process is an elastic transversal collision. $\mathcal{B}$ob's scattering angle α_i is determined by the trigonometric function for the triangle depicted in figure 10b.

□

The physical proof for the process in an elastic collision between three equivalent particles (21) contains the core for the proof of kinematical relations in an elastic collision between two generic objects (20). We generalize the configuration of Proposition 1 in two ways:

1. in the number $2 + 1$ of incident particles and

2. by refining scattering angle α_i after each (congruent) reservoir action w_T.

Proposition 2 *Consider an elastic collision of* $n{+}1$ *equivalent objects: One unit object* $①_{v_{(n)}}$ *comes in from left with initial velocity* $v_{(n)}$ *and a bundle of* n *unit objects* $①_{R_{\theta_1}v_{(1)}}, \dots, ①_{R_{\theta_n}v_{(1)}}$ *comes in from right with velocity* $v_{(1)}$

- *under orientations* $\theta_k := \frac{n+1}{2} \cdot \alpha_n - k \cdot \alpha_n$ *for* $k = 1, \dots, n$ [15]
- *ranging between* $\theta_1 = +\frac{\alpha_1}{2} - \frac{\alpha_n}{2}$ *and* $\theta_n = -\frac{\alpha_1}{2} + \frac{\alpha_n}{2}$ *with equal spacing* $\Delta\theta = \alpha_n$.

Velocities $v_{(1)}, v_{(n)}$ *and scattering angles* α_1, α_n *are chosen according to Lemma 1. In the final state the impulse of all objects is exactly reversed (see figure 11a)*

$$w: \quad ①_{v_{(n)}} \,,\, ①_{R_{\theta_1}v_{(1)}} \,, \dots ,\, ①_{R_{\theta_n}v_{(1)}} \;\Rightarrow\; ①_{-v_{(n)}} \,,\, ①_{-R_{\theta_1}v_{(1)}} \,, \dots ,\, ①_{-R_{\theta_n}v_{(1)}} \quad . \tag{28}$$

Proof: $\mathcal{A}$lice and $\mathcal{B}$ob construct a physical model analogous to Proposition 1. In **step I** they prepare congruent unit actions. They have access to an external reservoir of equivalent dynamical units $\{\mathcal{U}_\epsilon\big|_{\mathbf{0}}, ①_{\mathbf{0}}\}$. Against initially resting unit objects $①_{\mathbf{0}}$ they - temporarily - expend energy units $\mathcal{U}_\epsilon\,\big|_{\mathbf{0}}$ into suitable direction θ

$$w_\epsilon^{(\theta)}: \quad \mathcal{U}_\epsilon\big|_{\mathbf{0}} \,,\, ①_{\mathbf{0}} \,,\, ①_{\mathbf{0}} \;\Rightarrow\; ①_{\mathrm{R}_\theta(\epsilon\cdot\mathbf{v_1})} \,,\, ①_{\mathrm{R}_\theta(-\epsilon\cdot\mathbf{v_1})}$$

to *prepare* (congruent) transversal impulses $①_{\epsilon\cdot\mathbf{v_1}}$. They fire those reservoir particles $①_{\epsilon\cdot v_1}$ with velocity $\epsilon \cdot v_1$ into the way of incident object $①_{v_{(1)}}$ resp. $①_{v_{(n)}}$

$$w_T: \quad ①_{v_{(i)}} \,,\, ①_{\epsilon\cdot\mathbf{v_1}} \;\Rightarrow\; ①_{v'_{(i)}} \,,\, ①_{-\epsilon\cdot\mathbf{v_1}} \quad .$$

Each kick is an elastic transversal collision (see Lemma 1). $\mathcal{A}$lice and $\mathcal{B}$ob *provide* (congruent) elastic transversal collisions with various expedient orientations θ

$$w_T^{(\theta)} \;:=\; w_\epsilon^{(\theta)} * w_T \quad .$$

Each successively rotates direction of motion of incident object $①_{v_{(i)}}$ by angle α_i for $i = 1, n$.

In **step II** $\mathcal{A}$lice constructs the impulse reversion process $W_{(1)}$ separately for each element $①_{v_{(1)}}$ of the incident bundle. She associates a sequence of $N_{(1)} := \frac{180°}{\alpha_1}$ (transversal) actions

$$W_{(1)} \;:=\; w_T^{(-\frac{\alpha_1}{2}+\alpha_1)} * w_T^{(-\frac{\alpha_1}{2}+2\cdot\alpha_1)} * \dots * w_T^{(-\frac{\alpha_1}{2}+N_{(1)}\cdot\alpha_1)} \tag{29}$$

at same object $①_{v_{(1)}}$ which in each intermediate state moves freely with same velocity $v_{(1)}$.[16] Each (congruent) transversal impulse $①_{\epsilon\cdot v_1}$ turns incident object $①_{v_{(1)}}$ through another α_1.

---

[15] In the case of Proposition 1 with $n = 2$, $\alpha_1 = 60°$, $\alpha_2 = 30°$ we verify $\theta_1 := \frac{3}{2} \cdot \alpha_2 - \alpha_2 = \frac{1}{2} \cdot \alpha_2$ and $\theta_2 := \frac{3}{2} \cdot \alpha_2 - 2 \cdot \alpha_2 = -\frac{1}{2} \cdot \alpha_2$ in accordance with figure 8.

[16] We operate with inseparable units and their physical concatenation. In relativistic Kinematics {2.1} we operate with light clocks. We construct kinematical models e.g. $\mathbf{1} *_t \dots *_t \mathbf{1}^{(1)} *_s \dots *_s \mathbf{1}^{(n)} \;=_{t,s}\; \overline{\mathcal{A}_1\mathcal{O}}$ (8) by connecting those congruent kinematical units $\mathbf{1}$ in a consecutive $*_t$ resp. adjacent $*_s$ way. The way of concatenation of dynamical units is more subtle. Here we construct an impulse reversion process $w_T *^{(i)} w_T *^{(j)} \dots *^{(k)} w_T \sim_{E,\mathbf{p}} w$ which reproduces the direct elastic collision w with regard to energy and momentum. This physical model solely consists of congruent unit actions w_T from an external reservoir. We symbolize the way of their concatenation with the index $*^{(i)}$. In our superscript $^{(i)}$ we suppress a whole list of specifications e.g. in which particle, timing, position, etc. From illustrations analogous to figure 7 those characterizations are obvious. For the construction of impulse reversion process (29) it is sufficient to emphasize the spatial orientation θ in the coupling of each reservoir kick $w_T^{(\theta)}$.

After $N_{(1)}$ successive kicks its direction of motion is reversed. Similarly $\mathcal{B}$ob concatenates a sequence of $N_{(n)} := \frac{\pi}{\alpha_n}$ (transversal) actions

$$W_{(n)} \ := \ w_T^{(-\frac{\alpha_n}{2}+\alpha_n)} * w_T^{(-\frac{\alpha_n}{2}+2\cdot\alpha_n)} * \ldots * w_T^{(-\frac{\alpha_n}{2}+N_{(n)}\cdot\alpha_n)} \tag{30}$$

at same object $①_{v_{(n)}}$ which in each intermediate state moves freely with same velocity $v_{(n)}$. Each kick from reservoir particle $①_{\epsilon\cdot\mathbf{v_1}}$ turns incident object $①_{v_{(n)}}$ with velocity $v_{(n)}$ through corresponding angle α_n. After $N_{(n)}$ congruent kicks its direction of motion is reversed. The impulse of each incident object is reversed in a separate process.

In **step III** $\mathcal{A}$lice aligns her n impulse reversion processes $W_{(1)}$ for the n incident objects $①_{v_{(1)}}$ with the orientation of Bob's impulse reversion process $W_{(n)}$ for incident object $①_{v_{(n)}}$. In order to be able to recycle all - temporarily expended - dynamical units from the reservoir we require that both impulse reversion processes $W_{(1)}$ and $W_{(n)}$ match with one another. Both are built up from congruent transversal collisions $w_T^{(\theta)}$ by reservoir particle $①_{\epsilon\cdot\mathbf{v_1}}$ which kicks incident particles $①_{v_{(1)}}$ and $①_{v_{(n)}}$ around angle α_1 resp. α_n. We require that both scattering angles satisfy *matching condition*

$$\alpha_1 \stackrel{!}{=} n \cdot \alpha_n \ \ . \tag{31}$$

For fixed velocity $\epsilon\cdot\mathbf{v_1}$ of the reservoir particle $①_{\epsilon\cdot\mathbf{v_1}}$ and desired scattering angle α_i $\mathcal{A}$lice and $\mathcal{B}$ob have to compute and adjust the required initial velocities $v_{(i)}$ of their incident objects $①_{v_{(i)}}$ $i = 1, n$ according to Lemma 1.

$\mathcal{A}$lice rotates complete impulse reversion process (29) for first incident particle $①_{R_{\theta_1}v_{(1)}}$

$$\mathrm{R}_{\beta_1}\left[w_T^{(\vartheta_1)} * \ldots * w_T^{(\vartheta_{N_{(1)}})}\right] \ = \ w_T^{(\vartheta_1+\beta_1)} * \ldots * w_T^{(\vartheta_{N_{(1)}}+\beta_1)}$$

with $\vartheta_j := -\frac{\alpha_1}{2} + j\cdot\alpha_1$ for $j = 1, \ldots, N_{(1)}$ by an angle $\beta_1 := \pi + \underbrace{\frac{\alpha_1}{2} - \frac{\alpha_n}{2}}_{=:\theta_1}$. Similarly $\mathcal{A}$lice rotates impulse reversion process $\mathrm{R}_{\beta_k}\left[W_{(1)}\right]$ for every other element $①_{R_{\theta_k}v_{(1)}}$ of the incident bundle by an angle $\beta_k := \pi + \theta_k$ for $k = 1, \ldots, n$. Now $\mathcal{A}$lice and $\mathcal{B}$ob can *associate* their impulse reversion processes

$$W_{(n)} \ * \ \mathrm{R}_{\beta_1}\left[W_{(1)}\right] \ * \ \ldots \ * \ \mathrm{R}_{\beta_n}\left[W_{(1)}\right] \tag{32}$$

pairwise in diametrically opposed recoil particles $①_{\epsilon\cdot\mathbf{v_1}}$ and $①_{-\epsilon\cdot\mathbf{v_1}}$. The impulse reversion processes for individual objects $①_{v_{(1)}}$ and $①_{v_{(n)}}$ are *aligned* such that all byproducts from the preparation match one another at diametrically opposed locations. Along all dashed lines analogous to figure 8 they can recycle all their expended dynamical units back into the reservoir $\left\{\mathcal{U}_\epsilon\big|_{\mathbf{0}}, ①_{\mathbf{0}}\right\}$. After inserting in (32) all their congruent elastic transversal collisions

$$\left\{w_T^{(\gamma_1)} * \ldots * w_T^{(\gamma_{N_{(n)}})}\right\} \ * \ \left\{w_T^{(\vartheta_1+\beta_1)} * \ldots * w_T^{(\vartheta_{N_{(1)}}+\beta_1)}\right\} \\ * \ \ldots \ * \ \left\{w_T^{(\vartheta_1+\beta_n)} * \ldots * w_T^{(\vartheta_{N_{(1)}}+\beta_n)}\right\}$$

with $\gamma_l := -\frac{\alpha_n}{2} + l \cdot \alpha_n$ for $l = 1, \ldots, N_{(n)}$ they divide in pairs of antiparallel elastic transversal collisions

$$\left(w_T^{(\gamma_1)} * w_T^{(\delta_1+\beta_n)} \right) * \left(w_T^{(\gamma_2)} * w_T^{(\delta_1+\beta_{n-1})} \right) * \ldots * \left(w_T^{(\gamma_{N_{(n)}})} * w_T^{(\delta_{N_{(1)}}+\beta_1)} \right) \quad .$$

[17] Each associated tuple of four antiparallel recoil particles $①_{\epsilon \cdot \mathbf{v_1}}$, $①_{\epsilon \cdot \mathbf{v_1}}$, $①_{-\epsilon \cdot \mathbf{v_1}}$ and $①_{-\epsilon \cdot \mathbf{v_1}}$ reproduces the two - temporarily expended - congruent units of energy $\mathcal{U}_\epsilon \Big|_{\mathbf{0}}$ from step 1 and returns four resting particles back into the reservoir $\left\{ \mathcal{U}_\epsilon \Big|_{\mathbf{0}}, ①_{\mathbf{0}} \right\}$ (see figure 9c).

Replacement process (32) mediates the direct elastic collision - of $n+1$ incident objects $①_{v_{(1)}}, \ldots, ①_{v_{(1)}}, ①_{v_{(n)}}$ - by means of congruent unit actions $w_\epsilon^{(\theta)}$ with an external reservoir. In the end we have an *exact annihilation* of all expended dynamical units form the reservoir. Only the state of motion of all incident objects has changed. Their impulse is exactly reversed. As in step 3 of Proposition 1 this indirect impulse reversion process (32) and the direct elastic collision (28) are equivalent with regard to energy and momentum.

□

Our second modification of physical model (32) addresses the scattering angle α_i. For given $\alpha_1 = n \cdot \alpha_n$ the bundle of n unit objects $①_{R_{\theta_1} v_{(1)}}, \ldots, ①_{R_{\theta_n} v_{(1)}}$ comes in from right with fixed velocity $v_{(1)}$ under orientations ranging between $\theta_1 = +\frac{\alpha_1}{2} - \frac{\alpha_n}{2}$ and $\theta_n = -\frac{\alpha_1}{2} + \frac{\alpha_n}{2}$ (see Proposition 2). A *refinement* of each congruent transversal kick w_T (23) by reservoir objects $①_{\epsilon \cdot \mathbf{v_1}}$ against incident elements $①_{v_{(i)}}$ $i = 1, n$ changes the orientation of the matching bundle. In the limit $\epsilon \to 0$ the vanishing impetus of individual reservoir object $①_{\epsilon \cdot \mathbf{v_1}}$ kicks incident objects $①_{v_{(i)}}$ around vanishing scattering angles α_i. In the limit $\alpha_i \to 0$ the spreading of the bundle $\theta_1 - \theta_n := \alpha_1 - \alpha_n \overset{(31)}{=} (n-1) \cdot \alpha_n$ goes to zero. The bundle of n incident elements approximates a ray $\left\{ ①_{v_{(1)}}, \ldots, ①_{v_{(1)}} \right\} \equiv ⓝ_{v_{(1)}}$.

Proposition 3 *Our physical model (32) for an elastic collision between unit object* $①_{v_{(n)}}$ *and spreading bundle of n unit objects* $①_{R_{\theta_1} v_{(1)}}, \ldots, ①_{R_{\theta_n} v_{(1)}}$ *approximates by sufficient refinement*

- *of reservoir impulse units* $①_{\epsilon \cdot \mathbf{v_1}}$
- *for fixed velocity of the incident bundle* $v_{(1)}$

[17] Straight forward insertion confirms that first pair is aligned antiparallel and analogous for all the rest

$$\begin{aligned} \gamma_1 - (\delta_1 + \beta_n) \quad &=: \quad -\frac{\alpha_n}{2} + 1 \cdot \alpha_n \quad - \quad \left(-\frac{\alpha_1}{2} + 1 \cdot \alpha_1 \ + \ \pi + \frac{n+1}{2} \cdot \alpha_n - n \cdot \alpha_n \right) \\ &= \quad \frac{\alpha_n}{2} - \frac{\alpha_1}{2} - \pi + \frac{n \cdot \alpha_n}{2} - \frac{\alpha_n}{2} \overset{(31)}{=} -\pi \quad . \end{aligned}$$

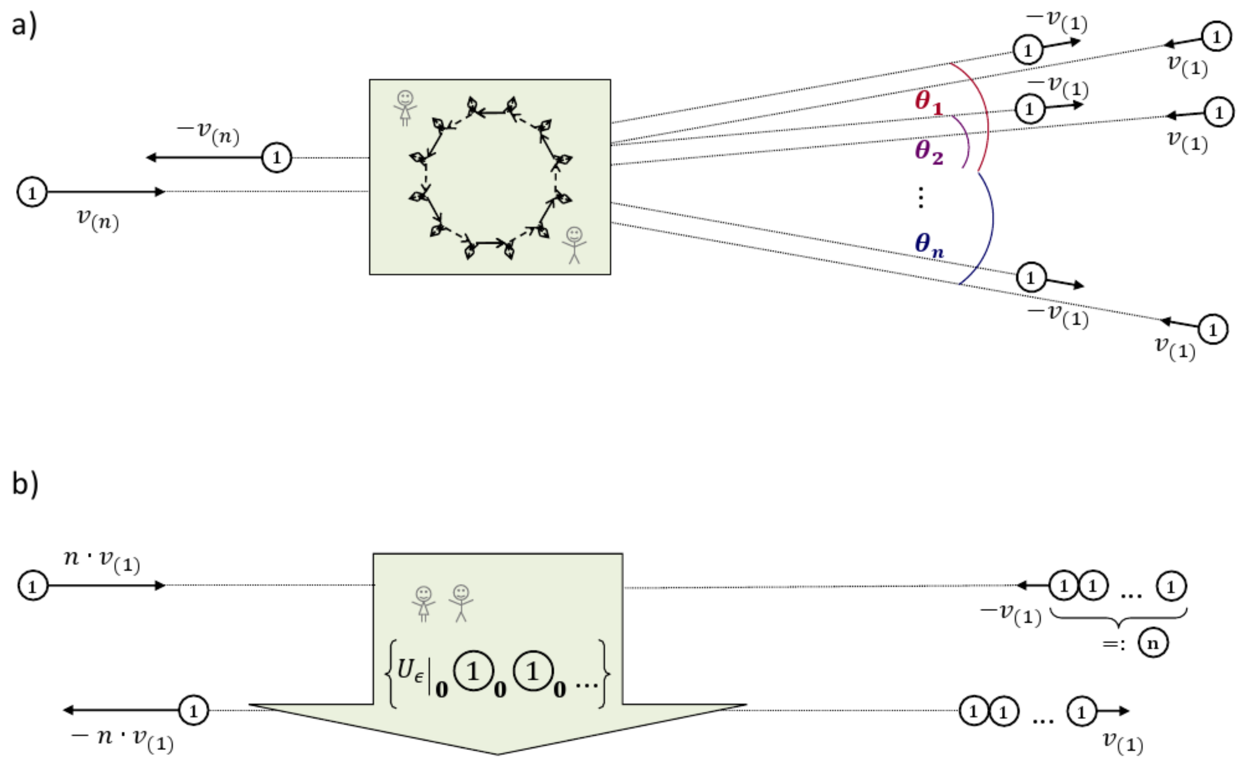

Figure 11: a) coarse grained perspective and b) in refinement limit bundle becomes a ray

$$\lim_{\epsilon \to 0} \quad W_{(n)} * \mathrm{R}_{\beta_1}\left[W_{(1)}\right] * \ldots * \mathrm{R}_{\beta_n}\left[W_{(1)}\right] \quad \sim_{E,\mathbf{p}} \quad w \tag{33}$$

the ideal of an elastic collision between different composites of equivalent objects (20)

$$w: \ \textcircled{n}_{1\cdot\mathbf{v}}\,,\ \textcircled{1}_{-n\cdot\mathbf{v}} \ \Rightarrow \ \textcircled{n}_{-1\cdot\mathbf{v}}\,,\ \textcircled{1}_{+n\cdot\mathbf{v}}$$

to any adjustable precision. The limit reproduces the elastic collision between a composite $\textcircled{n}_{\mathbf{v}_{(1)}}$ *with velocity* $\mathbf{v}_{(1)}$ *and a unit object* $\textcircled{1}_{\mathbf{v}_{(n)}}$ *with velocity* $\mathbf{v}_{(n)} = -n \cdot \mathbf{v}_{(1)}$ *(see figure 11).*

Proof: Let the velocity $v_{(1)} \overset{!}{=} v_{\mathbf{1}}$ of the n bundle elements $\textcircled{1}_{R_{\theta_1} v_{(1)}}, \ldots, \textcircled{1}_{R_{\theta_n} v_{(1)}}$ be fixed. $\forall \ \epsilon > 0$ $\mathcal{A}$lice and $\mathcal{B}$ob can provide a reservoir of congruent energy units $\mathcal{U}_\epsilon\big|_{\mathbf{0}}$ and impulse units $\textcircled{1}_{\epsilon\cdot\mathbf{v_1}}$ (22). Every transversal kick w_T (23) of reservoir particle $\textcircled{1}_{\epsilon\cdot\mathbf{v_1}}$ rotates bundle element $\textcircled{1}_{v_{(1)}}$ around corresponding angle $\alpha_1 := \alpha_1(v_{(1)}, \epsilon)$

$$\sin\left(\frac{\alpha_1}{2}\right) \overset{(27)}{=} \frac{\epsilon}{v_{(1)}} \quad .$$

$\mathcal{A}$lice constructs impulse reversion process $W_{(1)}$ for bundle element $\textcircled{1}_{v_{(1)}}$ by association of $N_{(1)} := \frac{180^\circ}{\alpha_1}$ (rational ratio corresponds to multiple revolutions) elastic transversal kicks (29).

According to matching condition every reservoir particle $①_{\epsilon\cdot\mathbf{v_1}}$ must kick unit object $①_{v_{(n)}}$ around corresponding angle $\alpha_n := \alpha_n(v_{(1)},\epsilon)$

$$\alpha_n \overset{(31)}{=} \frac{\alpha_1}{n} \quad .$$

$\mathcal{B}$ob constructs matching impulse reversion process $W_{(n)}$ for unit object $①_{v_{(n)}}$ by associating $N_{(n)} := n \cdot N_{(1)}$ congruent reservoir actions (30). The association of $\mathcal{A}$lice and $\mathcal{B}$ob's impulse reversion processes (32) mediates the elastic collision between one unit object $①_{v_{(n)}}$ with velocity $v_{(n)}$ and a bundle of n unit objects $①_{R_{\theta_1}v_{(1)}},\ldots,①_{R_{\theta_n}v_{(1)}}$ with fixed velocity $v_{(1)} \overset{!}{=} \mathbf{v_1}$ under orientations ranging between $\theta_1 = +\frac{\alpha_1}{2} - \frac{\alpha_n}{2}$ and $\theta_n = -\frac{\alpha_1}{2} + \frac{\alpha_n}{2}$ (see Proposition 2). Refinement $\epsilon \to 0$ of all reservoir actions w_ϵ (22) forces $\mathcal{A}$lice and $\mathcal{B}$ob to integrate an increasing number $N_{(i)}$ of congruent transversal collisions w_T into their physical model (32). In return the spreading of the bundle $\theta_1 - \theta_n := \alpha_1 - \alpha_n$

$$\lim_{\epsilon\to 0} \theta_1 - \theta_n \;=\; \lim_{\epsilon\to 0} \;(n-1)\cdot\alpha_1 \;=\; 0$$

narrows and the bundle approximates a ray.

Both impulse reversion processes $W_{(1)}$ and $W_{(2)}$ are built from congruent transversal kicks w_T by reservoir particle $①_{\epsilon\cdot v_1}$ against incident elements $①_{v_{(1)}}$ resp. $①_{v_{(n)}}$. Velocity $v_{(n)}$ is adjusted to fixed velocity $v_{(1)}$ and matching scattering angles α_1, α_n. We use matching condition $\alpha_1 \overset{!}{=} n\cdot\alpha_n$ in the form

$$\begin{aligned}\sin\left(\frac{\alpha_1}{2}\right) &\overset{!}{=} \sin\left(n\cdot\frac{\alpha_n}{2}\right)\\ &= \sum_{k=0}^{n-1}\begin{pmatrix} n \\ k\end{pmatrix}\cdot \underbrace{\cos^k\left(\frac{\alpha_n}{2}\right)}_{=\sqrt{1-\sin^2\left(\frac{\alpha_n}{2}\right)}^k}\cdot\sin^{n-k}\left(\frac{\alpha_n}{2}\right)\cdot\sin\left(\frac{1}{2}(n-k)\cdot\pi\right)\end{aligned}$$

where in the second step we use known trigonometric identity of multiple angles. According to Lemma 1 we can substitute $\sin\left(\frac{\alpha_n}{2}\right) = \frac{\epsilon}{v_{(i)}}$ for objects $i = 1, n$ and obtain

$$\begin{aligned}\frac{\epsilon}{v_{(1)}} &\overset{!}{=} \sum_{k=0}^{n-1}\begin{pmatrix} n \\ k\end{pmatrix}\cdot\sqrt{1-\frac{\epsilon^2}{v_{(n)}^2}}^k\cdot\left(\frac{\epsilon}{v_{(n)}}\right)^{n-k}\cdot\sin\left(\frac{1}{2}(n-k)\cdot\pi\right)\\ &= \left\{\sum_{k=0}^{n-2}\begin{pmatrix} n \\ k\end{pmatrix}\cdot\sqrt{1-\frac{\epsilon^2}{v_{(n)}^2}}^k\cdot\left(\frac{\epsilon}{v_{(n)}}\right)^{n-2-k}\cdot\sin\left(\frac{1}{2}(n-k)\cdot\pi\right)\right\}\cdot\frac{\epsilon^2}{v_{(n)}^2}\\ &\quad+\begin{pmatrix} n \\ n-1\end{pmatrix}\cdot\sqrt{1-\frac{\epsilon^2}{v_{(n)}^2}}^{n-1}\cdot\frac{\epsilon}{v_{(n)}}\cdot\sin\frac{\pi}{2}\end{aligned}\tag{34}$$

a well-defined relation $v_{(n)} := v_{(n)}(v_{(1)},\epsilon)$ $\forall$ fixed $v_{(1)} \overset{!}{=} \mathbf{v_1}$ and $\epsilon > 0$ $\exists$ matching $v_{(n)}(v_{(1)},\epsilon)$. Congruent transversal kicks w_T (23) from reservoir objects $①_{\epsilon\cdot\mathbf{v_1}}$ cause smaller scattering angle for unit object $①_{v_{(n)}}$ $\alpha_n < \alpha_1$ than for the bundle elements $①_{v_{(1)}}$. Hence its initial velocity is larger $v_{(n)} > v_{(1)} \stackrel{!}{=} v_{\mathbf{1}}$.

For small ϵ terms of linear or higher order in ϵ are neglected and relation (34) simplifies

$$\frac{1}{v_{(1)}} \stackrel{!}{=} \lim_{\epsilon \to 0} n \cdot \sqrt{1 - \frac{\epsilon^2}{v_{(n)}^2}}^{\,n-1} \cdot \frac{1}{v_{(n)}} = n \cdot \frac{1}{\lim_{\epsilon \to 0} v_{(n)}} .$$

In the refinement limit $\epsilon \to 0$ our physical model (32) mediates the elastic collision between a unit object $①_{v_{(n)}}$ with velocity

$$\lim_{\epsilon \to 0} v_{(n)} = n \cdot v_{(1)}$$

and a parallel beam of n elements $①_{v_{(1)}}, \ldots, ①_{v_{(1)}}$ with velocity $v_{(1)}$. In the initial and final state elements move with same velocity $\mathbf{v}_{(1)}$ - as if they were tightly connected in a composite $ⓝ_{\mathbf{v}_{(1)}}$ - exactly into opposite direction of incident object $①_{\mathbf{v}_{(n)}}$ with velocity $\mathbf{v}_{(n)} = -n \cdot \mathbf{v}_{(1)}$. This proves Proposition 3 and together with Propositions 1 and 2 also initial Theorem 1.

□

We learn something about elastic collisions which we did not presuppose before. We did not assume the kinematical relations (20) in an elastic collision between two generic objects. The arithmetic equation $m_1 \cdot \Delta v_1 = m_2 \cdot \Delta v_2$ in the mathematical formulation is physically justified. The effect of a direct elastic collision is reproduced by means of a physical model {3.1.3}. The key for the realization of the replacement process (26) is to mediate the direct elastic collision by means of congruent unit actions (23) from an external reservoir. Each is an elastic collision between equivalent objects ① and behaves in the same symmetrical way. The physical model solely consists of congruent unit actions. From the layout of those inseparable (!) unit actions we can specify the kinematic characterization. Now we know more about elastic collisions than before.

3.1.5 Quantification of Calorimeter Action

We illustrate the way how physicists construct experimental instruments to make energy and momentum measurable. We construct a physical model on the operation of a particle detector: a calorimeter

$$W_{\text{cal}} \quad \sim_{E,\mathbf{p}} \quad w$$

which reproduces the absorption action w of an individual object $\textcircled{a}\xrightarrow[v_a]{}$ in a calorimeter reservoir $\{\textcircled{1}_{\mathbf{0}}\}$. The participation of the physicist involves the act of *controlling* the process and *sorting* congruent unit actions $w_{\mathbf{1}}$.

An incoming particle $\textcircled{a}_{\mathbf{v}_a}$ will be slowed down $\textcircled{a}_{\mathbf{v}_a=0}$ in a cascade of successive collisions with those resting particles of the calorimeter. We set up a cascade of collisions where the process is controlled by a collective of physicists (see figure 2b). They control the preparation and coupling of many unit actions $w_{\mathbf{1}}$ such that the reservoir elements fly off in an organized way: As a result - of absorbing incident object $\textcircled{a}_{\mathbf{v}_a}$ - a certain number of reservoir elements $\sharp\,\{\textcircled{1}_{\mathbf{v_1}}\}$ will be knocked out of the calorimeter with standardized velocity $\mathbf{v_1}$ as well as a number of energy units $\sharp\,\{\mathbf{1}_E|_{\mathbf{0}}\}$.

We mediate the absorption action by an indirect replacement process. (A collective of) physicists set up and control the process in a series of elastic longitudinal collisions w_L (20). They couple incident object $\textcircled{a}_{\mathbf{v}_a}$ into a resting reservoir $\{\textcircled{1}_{\mathbf{0}}\}$. Their absorption <u>method</u> involves two steps:

1. They *place* resting reservoir particles $\textcircled{1}_{\mathbf{0}} * \ldots * \textcircled{1}_{\mathbf{0}}$ into the way of incident object $\textcircled{a}_{\mathbf{v}_a}$. They *generate* elastic longitudinal collisions w_L which

   - rebounds incident object $\textcircled{a}_{\mathbf{v}'_a}$ with reduced velocity and
   - kicks respective reservoir particles into unit motion $\textcircled{1}_{\mathbf{v_1}} * \ldots * \textcircled{1}_{\mathbf{v_1}}$

   (see figure 12b). Successively object $\textcircled{a}_{\mathbf{v}'_a}$ oscillates inside the deceleration cascade and kicks new initially resting particles out of the reservoir $\{\textcircled{1}_{\mathbf{0}}\}$ (see figure 13).

2. They *accumulate* and process those dynamical units $\{\textcircled{1}_{\mathbf{v_1}}\}$ which get kicked out on both sides of the calorimeter reservoir.

In the end incident object $\textcircled{a}_{\mathbf{v}'_a=0}$ comes to rest inside the calorimeter $\{\textcircled{1}_{\mathbf{0}}\}$. The absorption action w against incident object $\textcircled{a}_{\mathbf{v}_a}$ is specified by the replacement process W_{cal}. This physical model consists - on every level of the deceleration cascade - solely of congruent unit actions $w_{\mathbf{1}}$. In this model we can count the number of (equivalent) dynamical units $\sharp\,\{\mathbf{1}_E|_{\mathbf{0}}\}$ and $\sharp\,\{\mathbf{1}_{\mathbf{p}}\}$ which get extracted from the calorimeter reservoir $\{\textcircled{1}_{\mathbf{0}}\}$. In this way the absorption action w of incident object $\textcircled{a}_{\mathbf{v}_a}$ in a calorimeter becomes measurable. By means of our physical model W_{cal} the energy E_a and momentum $\mathbf{p}_a$ of incident particle $\textcircled{a}_{\mathbf{v}_a}$ is metricized. We illustrate the (controlled) absorption method for a simple configuration.

Proposition 4 *Alice provides a reservoir with* 25 *resting unit objects* $\{\textcircled{1}_{\mathbf{0}}\}$. *One incident unit object* $\textcircled{1}_{10\cdot\mathbf{v_1}}$ *with velocity* $10\cdot\mathbf{v_1}$ *comes to rest* $\textcircled{1}_{\mathbf{0}}$ *after a (controlled) cascade of elastic longitudinal collisions* W_{cal} *with the reservoir*

$$W_{\text{cal}}:\quad \textcircled{1}_{10\cdot\mathbf{v_1}}\,,\ 25\cdot\textcircled{1}_{\mathbf{0}}\quad\Rightarrow\quad \textcircled{1}_{\mathbf{0}}\,,\ 10\cdot\{\textcircled{1}_{2\cdot\mathbf{v}},\textcircled{1}_{-2\cdot\mathbf{v}}\}\ ,\ 5\cdot\textcircled{1}_{2\cdot\mathbf{v_1}}\quad .$$

After its absorption an (organized) output

- 10 *pairs of antiparallel recoil particles* $\{\textcircled{1}_{2\cdot\mathbf{v}},\textcircled{1}_{-2\cdot\mathbf{v}}\}$ - *each represents congruent energy unit* $\mathcal{U}_2\big|_{\mathbf{0}}$ - *and*
- 5 *congruent impulse units* $\textcircled{1}_{2\cdot\mathbf{v_1}}$

is extracted from $\mathcal{A}$*lice external reservoir* $\{\textcircled{1}_{\mathbf{0}}\}$ *(see figure 13).*

Proof: In **step I** $\mathcal{A}$lice successively decelerates incident object $\textcircled{1}_{10\cdot\mathbf{v_1}}$ by a cascade of elastic collisions with resting particles from the reservoir $\{\textcircled{1}_{\mathbf{0}}\}$. $\mathcal{A}$lice prepares a composite of n unit objects from her reservoir $\underbrace{\textcircled{1}_{\mathbf{0}}*\ldots*\textcircled{1}_{\mathbf{0}}}_{n\times}=:\textcircled{n}_{\mathbf{0}}$. If for $\mathcal{B}$ob unit object $\textcircled{1}_{n\cdot\mathbf{v_1}}$ comes in from left with velocity $n\cdot\mathbf{v_1}$ and composite $\textcircled{n}_{-\mathbf{v_1}}$ comes in from right with velocity $-\mathbf{v_1}$ they interact in an elastic longitudinal collision

$$w_L:\quad \textcircled{1}_{n\cdot\mathbf{v_1}}\,,\ \textcircled{n}_{-\mathbf{v_1}}\quad\overset{(20)}{\Rightarrow}\quad \textcircled{1}_{-n\cdot\mathbf{v_1}}\,,\ \textcircled{n}_{\mathbf{v_1}}$$

(see figure 12a). Let $\mathcal{B}$ob move relative to $\mathcal{A}$lice with constant velocity $\mathbf{v}_{\mathcal{B}}=1\cdot\mathbf{v}_{\mathbf{1}(\mathcal{A})}$ to the right. For $\mathcal{A}$lice measured values of motion $\mathbf{v}_i^{(\mathcal{A})}=\mathbf{v}_i^{(\mathcal{B})}+\mathbf{v}_{\mathcal{B}}^{(\mathcal{A})}$ transform - Galilei covariant - by vectorial addition for both objects $i=1,n$ (see Remark 8). For $\mathcal{A}$lice incident object $\textcircled{1}_{(n+1)\cdot\mathbf{v_1}}$ collides into a composite of n resting units $\textcircled{n}_{\mathbf{0}}$. She will see an elastic longitudinal collision into the *right* side of the calorimeter

$$w_L^{(r)}:\quad \textcircled{1}_{(n+1)\cdot\mathbf{v_1}}\,,\ \textcircled{n}_{\mathbf{0}}\quad\Rightarrow\quad \textcircled{1}_{-(n-1)\cdot\mathbf{v_1}}\,,\ \textcircled{n}_{2\cdot\mathbf{v_1}} \tag{35}$$

(see figure 12b). The object $\textcircled{1}_{-(n-1)\cdot\mathbf{v_1}}$ rebounds antiparallel with reduced velocity $(n-1)\cdot v_{\mathbf{1}}$ to the left. On the left side $\mathcal{A}$lice places a new composite of $n-2$ unit objects from her reservoir $\underbrace{\textcircled{1}_{\mathbf{0}}*\ldots*\textcircled{1}_{\mathbf{0}}}_{(n-2)\times}=:(n-2)\cdot\textcircled{1}_{\mathbf{0}}$ and generates the next elastic longitudinal collision against the *left* side of the calorimeter

$$w_L^{(l)}:\quad \textcircled{1}_{-((n-2)+1)\cdot\mathbf{v_1}}\,,\ (n-2)\cdot\textcircled{1}_{\mathbf{0}}\quad\Rightarrow\quad \textcircled{1}_{(n-3)\cdot\mathbf{v_1}}\,,\ (n-2)\cdot\textcircled{1}_{-2\cdot\mathbf{v_1}}\quad . \tag{36}$$

$\mathcal{A}$lice associates the two elastic longitudinal collisions (35) and (36)

$$W\ :=\ w_L^{(r)}*w_L^{(l)} \tag{37}$$

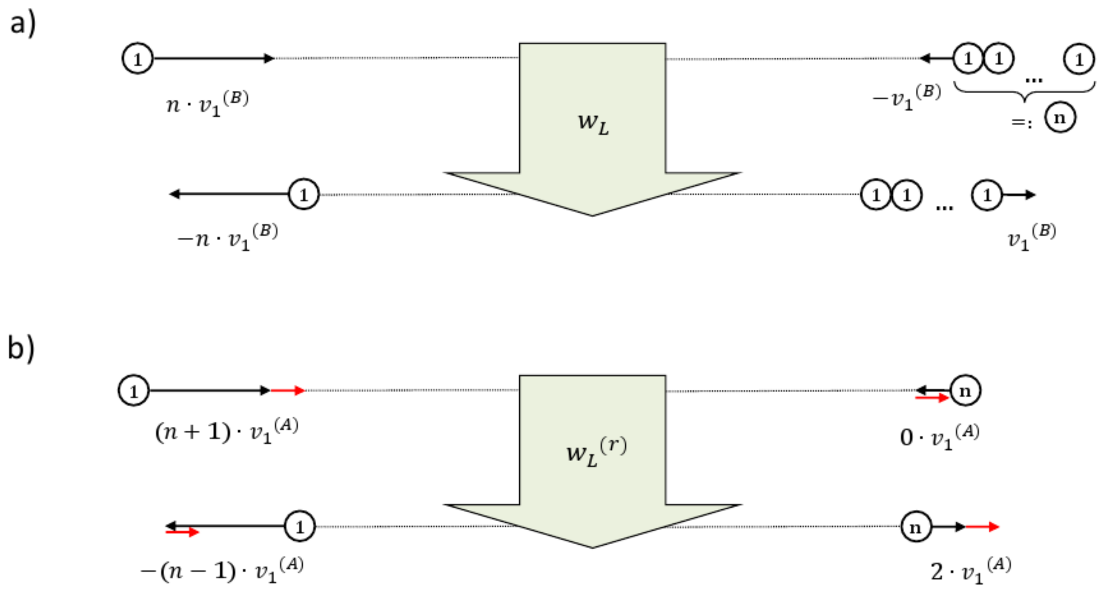

Figure 12: a) elastic longitudinal collision w_L for $\mathcal{B}ob$ and b) covariant transformation of same process to perspective of $\mathcal{A}lice$

at same object $①_{-(n-1)\cdot \mathbf{v_1}}$ which in between right and left collision moves freely with same velocity $-(n-1)\cdot \mathbf{v_1}$. After each round of right and left collisions $w_L^{(r)} * w_L^{(l)}$ incident object $①_{(n+1)\cdot \mathbf{v_1}}$ slows down

$$W: \quad ①_{(n+1)\cdot \mathbf{v_1}}\,,\ (n-2)\cdot ①_{\mathbf{0}}\,,\ n\cdot ①_{\mathbf{0}} \quad \Rightarrow \quad ①_{(n-3)\cdot \mathbf{v_1}}\,,\ \underbrace{(n-2)\cdot ①_{-2\cdot \mathbf{v_1}}\,,\ n\cdot ①_{2\cdot \mathbf{v_1}}}_{(n-2)\cdot \{①_{-2\cdot \mathbf{v}}, ①_{2\cdot \mathbf{v}}\}\,,\ 2\cdot ①_{2\cdot \mathbf{v_1}}} \tag{38}$$

In return a certain number of initially resting particles will be knocked out of the reservoir $\{①_{\mathbf{0}}\}$ with same velocity $2\cdot v_1$: On the left side $n-2$ elements $①_{-2\cdot \mathbf{v_1}}$ and on the right side $(n-2)+2$ elements $①_{2\cdot \mathbf{v_1}}$ (see figure 13).

In **step II** $\mathcal{A}lice$ processes those dynamical units $\{①_{2\cdot \mathbf{v_1}}\}$ which get kicked out on both sides of the calorimeter reservoir. Each pair of antiparallel recoil particles $\{①_{-2\cdot \mathbf{v}}, ①_{2\cdot \mathbf{v}}\}$ in (38) can be recycled by means of unit action (22)

$$w_2^{-1}: \quad ①_{-2\cdot \mathbf{v_1}}\,,\ ①_{2\cdot \mathbf{v_1}} \ \Rightarrow\ \mathcal{U}_2\big|_{\mathbf{0}}\,,\ ①_{\mathbf{0}}\,,\ ①_{\mathbf{0}} \ . \tag{39}$$

Both resting objects $①_{\mathbf{0}}$ return back into the calorimeter reservoir $\{①_{\mathbf{0}}\}$. From each round of right and left collisions W incident object $①_{(n+1)\cdot \mathbf{v_1}} \Rightarrow ①_{(n-3)\cdot \mathbf{v_1}}$ slows down (38). $\mathcal{A}lice$ extracts the calorimeter *reservoir balance* for deceleration

$$\mathrm{RB}\left[①_{(n+1)\cdot \mathbf{v_1}} \Rightarrow ①_{(n-3)\cdot \mathbf{v_1}}\right] \ = \ (n-2)\cdot \mathcal{U}_2\big|_{\mathbf{0}} \ ,\ \ 2\cdot ①_{2\cdot \mathbf{v_1}} \tag{40}$$

of $n-2$ congruent units of energy $\mathcal{U}_2\big|_{\mathbf{0}}$ and 2 congruent units of momentum $①_{2\cdot\mathbf{v_1}}$.

For incident object $①_{10\cdot\mathbf{v_1}}$ $\mathcal{A}$lice conducts elastic reversion sequence $W := w_L^{(r)} * w_L^{(l)}$ (38) two consecutive times

$$W_{\text{cal}} \;:=\; W * W$$

until it comes to rest (see figure 13). Then she can count congruent measurement units: $7+3$ units of energy $\mathcal{U}_2\big|_{\mathbf{0}}$ and $2+2+1$ impulse units $①_{2\cdot\mathbf{v_1}}$ from both rounds of her calorimeter-collision-cascade W_{cal}.[18]

□

Proposition 5 *$\mathcal{A}$lice provides a calorimeter reservoir with resting unit objects $\{①_{\mathbf{0}}\}$. The calorimeter-collision-cascade W_{cal} is a physical model for the absorption action of unit object $①_{n\cdot\mathbf{v_1}}$ with velocity $n\cdot\mathbf{v_1}$ in her calorimeter where it comes to rest $①_{\mathbf{0}}$*

$$W_{\text{cal}}: \quad ①_{n\cdot\mathbf{v_1}}\,,\ \{①_{\mathbf{0}}\} \quad \Rightarrow \quad ①_{\mathbf{0}}\,,\ \text{RB}\left[①_{n\cdot\mathbf{v_1}} \Rightarrow ①_{\mathbf{0}}\right] \quad .$$

This reversible process is characterized - in an observer independently reproducible way - by its output: In the reservoir balance for the absorption

$$\text{RB}\left[①_{n\cdot\mathbf{v_1}} \Rightarrow ①_{\mathbf{0}}\right] \;=\; \left(\frac{1}{2}\cdot n^2 - \frac{1}{2}\cdot n\right)\cdot \mathbf{1}_E\big|_{\mathbf{0}}\ ,\quad n\cdot\mathbf{1_p} \tag{41}$$

a certain number of congruent units of energy $\mathbf{1}_E\big|_{\mathbf{0}}$ and momentum $\mathbf{1_p} := ①_{\mathbf{v_1}}$ are extracted from $\mathcal{A}$lice external reservoir $\{①_{\mathbf{0}}\}$.

---

[18]Prob: For the example in figure 13 the reservoir balance of extracted (congruent) dynamical units

$$\text{RB}\left[①_{10\cdot v_1} \Rightarrow ①_{\mathbf{0}}\right] \;=\; (7+3)\cdot\mathcal{U}_2\big|_{\mathbf{0}}\ ,\ (2+2+1)\cdot ①_{2\cdot\mathbf{v_1}}$$

is consistent with not yet justified equations '$E=\frac{m}{2}\cdot v^2$' and '$p=m\cdot v$'. When applied to momentum units $①_{2\cdot\mathbf{v_1}}$ and energy units $\mathcal{U}_2\big|_{\mathbf{0}}$ (39)

$$\begin{array}{lcl} E\left[①_{2\cdot\mathbf{v_1}}\right] &=& \frac{1}{2}\cdot 1\cdot 2^2 \\ \mathbf{p}\left[①_{2\cdot\mathbf{v_1}}\right] &=& 1\cdot 2 \end{array} \qquad \begin{array}{lcl} E\left[\mathcal{U}_2\big|_{\mathbf{0}}\right] &=& E\left[①_{2\cdot\mathbf{v_1}}\,,\ ①_{-2\cdot\mathbf{v_1}}\right] \;=\; 2\cdot\left(\frac{1}{2}\cdot 1\cdot 2^2\right) \\ \mathbf{p}\left[\mathcal{U}_2\big|_{\mathbf{0}}\right] &=& \mathbf{p}\left[①_{2\cdot\mathbf{v_1}}\,,\ ①_{-2\cdot\mathbf{v_1}}\right] \;=\; 0 \end{array}$$

insertion into reservoir balance

$$\begin{aligned} E\left[①_{10\cdot v_1}\right] &= (7+3)\cdot E\left[\mathcal{U}_2\big|_{\mathbf{0}}\right] \;+\; (2+2+1)\cdot E\left[①_{2\cdot\mathbf{v_1}}\right] \\ &= 10\cdot 4 \;+\; 5\cdot 2 \;=\; 50 \;=\; \frac{1}{2}\cdot 1\cdot 10^2 \\ \\ p\left[①_{10\cdot v_1}\right] &= (7+3)\cdot p\left[\mathcal{U}_2\big|_{\mathbf{0}}\right] \;+\; (2+2+1)\cdot p\left[①_{2\cdot\mathbf{v_1}}\right] \\ &= 0 \;+\; 5\cdot 2 \;=\; 10 \;=\; 1\cdot 10\ . \end{aligned}$$

confirms familiar equations $E=\frac{m}{2}\cdot v^2$ and $p=m\cdot v$ - which were not postulated in our physical approach.

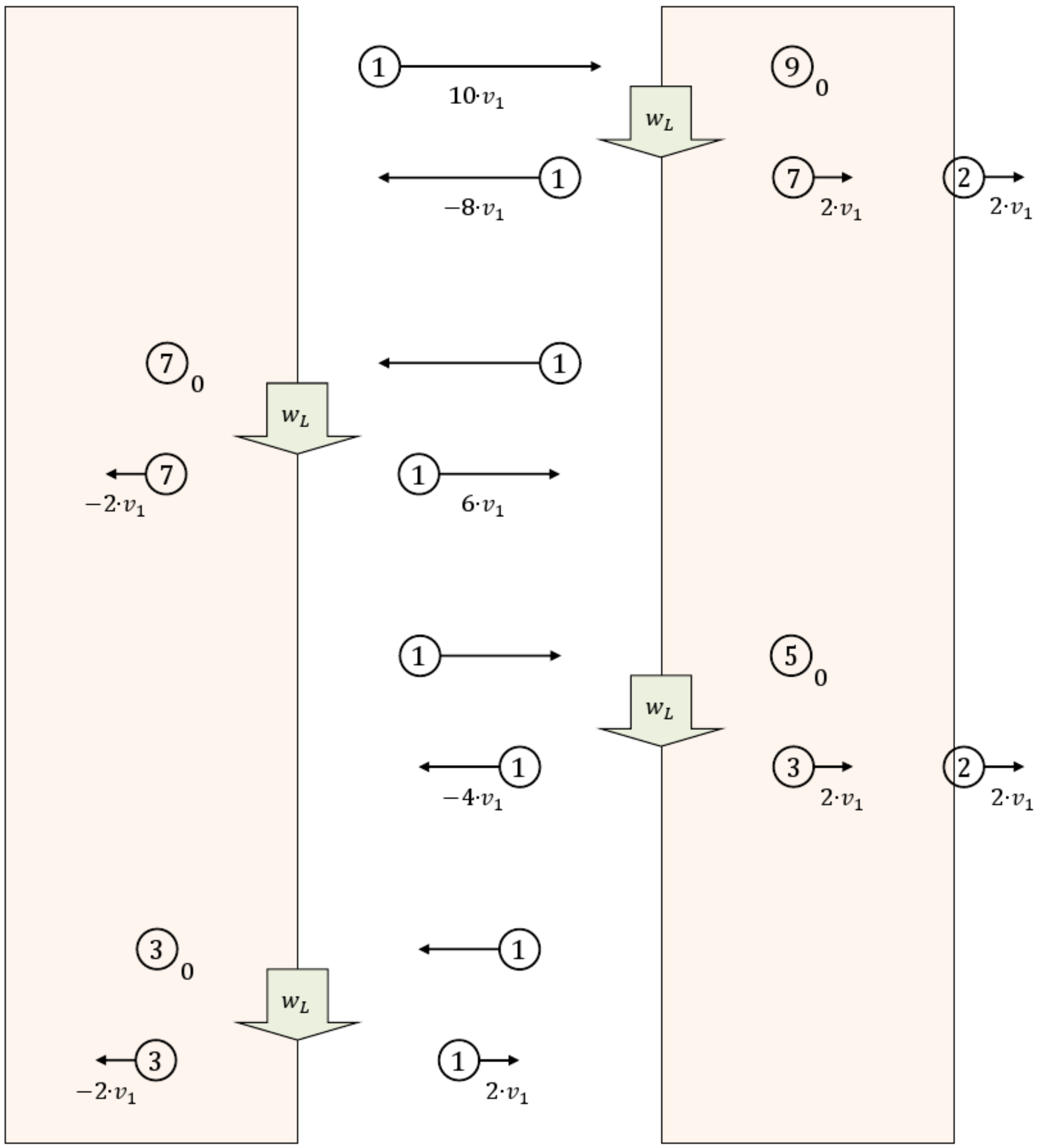

Figure 13: incident particle successively comes to rest by means of elastic collisions with initially resting particles on the left resp. right side of the calorimeter reservoir $\left\{ \textcircled{1} \big|_{\mathbf{v=0}} \right\}$

Proof: Let incident unit object $①_{n\cdot\mathbf{v_1}}$ come in from left with velocity $n\cdot\mathbf{v_1} = (2N+1)\cdot 2\cdot\mathbf{v_1}$ $N \in \mathbb{N}$. $\mathcal{A}$lice has to conduct the cycle of (elastic) collisions into the left and right side of the calorimeter N consecutive times

$$W_{\text{cal}} := W^{(1)} * \cdots * W^{(N)} \tag{42}$$

to bring incident object $①_{(4N+2)\cdot\mathbf{v_1}}$ to rest. Then she can count congruent measurement units for energy $\mathcal{U}_2\big|_{\mathbf{0}}$ and momentum $①_{2\cdot\mathbf{v_1}}$.

After every cycle $i = 1,\ldots,N$ of right and left collisions $W^{(i)} := w_L^{(r)} * w_L^{(l)}$ incident object $①_{(4i+2)\cdot\mathbf{v_1}} \Rightarrow ①_{(4i-2)\cdot\mathbf{v_1}}$ slows down (38). As a result $\mathcal{A}$lice extracts the corresponding reservoir balance of deceleration

$$\text{RB}\left[①_{(4i+2)\cdot\mathbf{v_1}} \Rightarrow ①_{(4i-2)\cdot\mathbf{v_1}}\right] \overset{(40)}{=} (4i-1)\cdot\mathcal{U}_2\big|_{\mathbf{0}} \ , \ 2\cdot①_{2\cdot\mathbf{v_1}} \tag{43}$$

out of the calorimeter reservoir $\{①_{\mathbf{0}}\}$. On each level $W^{(i)}$ $i = 1,\ldots,N$ of the physical model for the absorption W_{cal} $\mathcal{A}$lice extracts equivalent units of energy $\mathcal{U}_2\big|_{\mathbf{0}}$ and momentum $①_{2\cdot\mathbf{v_1}}$ (see figure 13). By the *congruence principle* $\mathcal{A}$lice extracts the reservoir balance of absorption

$$\begin{aligned}
\text{RB}\left[①_{(2N+1)\cdot 2\cdot\mathbf{v_1}} \Rightarrow ①_{\mathbf{0}}\right] &= \text{RB}\left[①_{(4N+2)\cdot\mathbf{v_1}} \Rightarrow ①_{2\cdot\mathbf{v_1}}\right] + \text{RB}\left[①_{2\cdot\mathbf{v_1}} \Rightarrow ①_{\mathbf{0}}\right] \\
&= \sum_{i=1}^{N} \text{RB}\left[①_{(4i+2)\cdot\mathbf{v_1}} \Rightarrow ①_{(4i-2)\cdot\mathbf{v_1}}\right] + \text{RB}\left[①_{2\cdot\mathbf{v_1}} \Rightarrow ①_{\mathbf{0}}\right] \\
&\overset{(43)}{=} \sum_{i=1}^{N} \left((4i-1)\cdot\mathcal{U}_2\big|_{\mathbf{0}} \, , \, 2\cdot①_{2\cdot\mathbf{v_1}}\right) + \left(0\cdot\mathcal{U}_2\big|_{\mathbf{0}} \, , \, 1\cdot①_{2\cdot\mathbf{v_1}}\right) \\
&= \underbrace{(2\cdot N^2 + N)}_{\frac{1}{2}\cdot(2N+1)^2 - \frac{1}{2}\cdot(2N+1)} \cdot\mathcal{U}_2\big|_{\mathbf{0}} \ , \ (2N+1)\cdot①_{2\cdot\mathbf{v_1}}
\end{aligned} \tag{44}$$

from the calorimeter-collision-cascade.

Similarly let incident object $①_{n\cdot\mathbf{v_1}}$ come in from left with velocity $n\cdot\mathbf{v_1} = (2N+2)\cdot 2\cdot\mathbf{v_1}$ $N \in \mathbb{N}$. $\mathcal{A}$lice controls the exactly elastic reversion sequence $W^{(i)}$ N consecutive times $W_{\text{cal}} := W^{(1)} * \cdots * W^{(N)}$ until incident object $①_{(4i+4)\cdot\mathbf{v_1}} \Rightarrow ①_{\mathbf{0}}$ comes to rest. $\mathcal{A}$lice extracts the reservoir balance of absorption

$$\begin{aligned}
\text{RB}\left[①_{(2N+2)\cdot 2\cdot\mathbf{v_1}} \Rightarrow ①_{\mathbf{0}}\right] &= \text{RB}\left[①_{(4N+4)\cdot\mathbf{v_1}} \Rightarrow ①_{-(4N+2)\cdot\mathbf{v_1}}\right] + \text{RB}\left[①_{-(2N+1)\cdot 2\cdot\mathbf{v_1}} \Rightarrow ①_{\mathbf{0}}\right] \\
&\overset{(35)(44)}{=} (4N+3)\cdot①_{2\cdot\mathbf{v_1}} \ , \ (2\cdot N^2+N)\cdot\mathcal{U}_2\big|_{\mathbf{0}} \ , \ (2N+1)\cdot①_{-2\cdot\mathbf{v_1}} \\
&\overset{(39)}{=} \underbrace{(2\cdot N^2 + N \quad + \quad 2N+1)}_{\frac{1}{2}\cdot(2N+2)^2 - \frac{1}{2}\cdot(2N+2)} \cdot\mathcal{U}_2\big|_{\mathbf{0}} \ , \ (2N+2)\cdot①_{2\cdot\mathbf{v_1}}
\end{aligned}$$

where in the last step we recycle $2N+1$ pairs of antiparallel recoil particles $\{①_{-2\cdot\mathbf{v_1}}, ①_{2\cdot\mathbf{v_1}}\}$ back into the calorimeter reservoir $\{①_{\mathbf{0}}\}$ in return for $2N+1$ extra units of energy $\mathcal{U}_2\big|_{\mathbf{0}}$. To simplify notation let $\mathcal{A}$lice use different physical units $\mathbf{1}^{(\mathcal{A})} \not\approx \mathbf{1}$ with velocity $\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}} := 2 \cdot \mathbf{v_1}$ and dynamical units $\mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}} := \mathcal{U}_2\big|_{\mathbf{0}}$ and $\mathbf{1}_{\mathbf{p}}^{(\mathcal{A})} := (1)_{\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}}}$ in accordance with her fixed unit action $w_{\mathbf{1}^{(\mathcal{A})}}$ (17). In the case with even and odd number of reservoir reflections $\mathcal{A}$lice obtains the same reservoir balance for absorption in a calorimeter (41).

The calorimeter-collision-cascade provides an elastic (practically instantaneous) sequence of actions in system $(a)_{n\cdot\mathbf{v_1}} \cup \{(1)_{\mathbf{0}}\}$ of incident particle and $\mathcal{A}$lice calorimeter reservoir. Every step in the controlled cascade of elastic collisions is reversible. In the final state the incident particle comes to rest $(a)_{n\cdot\mathbf{v_1}} \Rightarrow (a)_{\mathbf{0}}$ while a number

- $k_a^{(\mathcal{A})} := \frac{1}{2} \cdot n^2 - \frac{1}{2} \cdot n$ of congruent energy units $\mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}}$ and
- $l_a^{(\mathcal{A})} := n$ of congruent impulse units $\mathbf{1}_{\mathbf{p}}^{(\mathcal{A})} = (1)_{\mathbf{v}_{\mathbf{1}}^{(\mathcal{A})}}$

is extracted from $\mathcal{A}$lice calorimeter reservoir $\{(1)_{\mathbf{0}}\}$.

□

Remark 9 *The calorimeter-collision-cascade W_{cal} (42) is a physical model for the absorption action w of incident particle $(a)_{n\cdot\mathbf{v_1}}$ in a calorimeter*

$$W_{\mathrm{cal}} \quad \sim_{E,\mathbf{p}} \quad w \ .$$

In $\mathcal{A}$lice (controlled) replacement process W_{cal} the particle $(a)_{n\cdot\mathbf{v_1}}$ is absorbed - in a reversible and practically instantaneous way - in return for the extraction of equivalent dynamical units $\mathbf{1}_E\big|_{\mathbf{0}}$ and $\mathbf{1}_{\mathbf{p}}$ from the reservoir. Every level of the successive deceleration cascade solely consists of congruent unit actions $w_{\mathbf{1}}$.

Definition 7 *The <u>reservoir balance</u> of absorption*

$$\mathrm{RB}^{(\mathcal{A})}\left[(a)_{\overrightarrow{\mathbf{v}_a}} \ \Rightarrow (a)_{\mathbf{0}}\right] \ := \ k_a^{(\mathcal{A})} \cdot \mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}} \ , \ \ l_a^{(\mathcal{A})} \cdot \mathbf{1}_p^{(\mathcal{A})} \tag{45}$$

is the extracted output from the calorimeter-collision-cascade W_{cal}. When $\mathcal{A}$lice expends - in an organized way - $k_a^{(\mathcal{A})}$ energy units $\mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}}$ and $l_a^{(\mathcal{A})}$ impulse units $\mathbf{1}_{\mathbf{p}}^{(\mathcal{A})}$ against $m_a^{(\mathcal{A})}$ resting elements from her calorimeter reservoir $\{(1)_{\mathbf{0}}\}$ she can construct a physical model

$${W_{\mathrm{cal}}}^{-1}: \ \ k_a^{(\mathcal{A})} \cdot \mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}} \ , \ \ l_a^{(\mathcal{A})} \cdot \mathbf{1}_{\mathbf{p}}^{(\mathcal{A})} \ , \ \ m_a^{(\mathcal{A})} \cdot (1)_{\mathbf{0}} \ \ \Rightarrow \ \ (1)_{\mathbf{v}_a} * \ldots * (1)_{\mathbf{v}_a}$$

which reproduces energy and impulse of incident object $(a)_{\overrightarrow{\mathbf{v}_a}}$

$$(a)_{\mathbf{v}_a} \quad \sim_{E,\mathbf{p}} \quad (1)_{\mathbf{v}_a} * \ldots * (1)_{\mathbf{v}_a} \ .$$

The calorimeter-collision-cascade W_{cal} reproduces an absorption by elastic redistribution of dynamical units. The output of that elastic redistribution has various properties which are essential for our metrization.

Lemma 2 *The reservoir balance for absorption of a system of multiple elements* $\textcircled{\scriptsize i}_{\mathbf{v}_i}$ *with* $i = 1, \ldots, n$ *is* <u>*additive*</u> *in the number of congruent dynamical units*

$$\mathrm{RB}\left[\textcircled{\scriptsize 1}_{\mathbf{v}_1}, \ldots, \textcircled{\scriptsize n}_{\mathbf{v}_n} \Rightarrow \textcircled{\scriptsize 1}_{\mathbf{0}}, \ldots, \textcircled{\scriptsize n}_{\mathbf{0}}\right] \;=\; \mathrm{RB}\left[\textcircled{\scriptsize 1}_{\mathbf{v}_1} \Rightarrow \textcircled{\scriptsize 1}_{\mathbf{0}}\right] \;+ \ldots +\; \mathrm{RB}\left[\textcircled{\scriptsize n}_{\mathbf{v}_n} \Rightarrow \textcircled{\scriptsize n}_{\mathbf{0}}\right] \quad .$$

For a generic change in the state of motion it decomposes

$$\mathrm{RB}\left[\textcircled{\scriptsize 1}_{\mathbf{v}_1}, \ldots, \textcircled{\scriptsize n}_{\mathbf{v}_n} \Rightarrow \textcircled{\scriptsize 1}_{\mathbf{v}'_1}, \ldots, \textcircled{\scriptsize n}_{\mathbf{v}'_n}\right] \;=\; \sum_{i=1}^{n} \left(\mathrm{RB}\left[\textcircled{\scriptsize i}_{\mathbf{v}_i} \Rightarrow \textcircled{\scriptsize i}_{\mathbf{0}}\right] \,-\, \mathrm{RB}\left[\textcircled{\scriptsize n}_{\mathbf{v}'_i} \Rightarrow \textcircled{\scriptsize i}_{\mathbf{0}}\right]\right) \tag{46}$$

Proof: $\mathcal{A}$lice can absorb every element $\textcircled{\scriptsize i}_{\mathbf{v}_i} \Rightarrow \textcircled{\scriptsize i}_{\mathbf{0}}$ individually in a separate calorimeter-collision-cascade $W^{(i)}_{\mathrm{cal}}$. By the congruence of extracted dynamical units $\mathbf{1}_E\big|_{\mathbf{0}}$ and $\mathbf{1}_{\mathbf{p}}$ in those physical models $W^{(1)}_{\mathrm{cal}}, \ldots, W^{(n)}_{\mathrm{cal}}$ the total number of extracted measurement units of energy $\mathbf{1}_E\big|_{\mathbf{0}}$ resp. momentum $\mathbf{1}_{\mathbf{p}}$ simply adds up. Finally by expending the reservoir balance of absorption $-\mathrm{RB}\left[\textcircled{\scriptsize a}_{\mathbf{v}'_a} \Rightarrow \textcircled{\scriptsize a}_{\mathbf{0}}\right]$ against resting reservoir object $\textcircled{\scriptsize a}_{\mathbf{0}}$ in an organized way the reversible replacement process ${W_{\mathrm{cal}}}^{-1}$ reproduces the initial state of motion of object $\textcircled{\scriptsize a}$

$${W_{\mathrm{cal}}}^{-1} : \;\; \textcircled{\scriptsize a}_{\mathbf{0}}\, ,\, \mathrm{RB}\left[\textcircled{\scriptsize a}_{\mathbf{v}'_a} \Rightarrow \textcircled{\scriptsize a}_{\mathbf{0}}\right] \;\;\Rightarrow\;\; \textcircled{\scriptsize a}_{\mathbf{v}'_a}\, ,\, \{\textcircled{\scriptsize 1}_{\mathbf{0}}\} \quad .$$

□

Let $\mathcal{A}$lice and $\mathcal{B}$ob share a calorimeter reservoir $\{\textcircled{\scriptsize 1}_{\mathbf{0}}\}$ with equivalent unit objects $\textcircled{\scriptsize 1}^{(\mathcal{A})} \sim_{m^{(\mathrm{inert})}} \textcircled{\scriptsize 1}^{(\mathcal{B})}$. $\mathcal{A}$lice controls a calorimeter-collision-cascade ${W_{\mathrm{cal}}}^{(\mathcal{A})}$ which solely consists of congruent unit actions $w_{\mathbf{1}^{(\mathcal{A})}} \;:\; \mathbf{1}^{(\mathcal{A})}_E\big|_{\mathbf{0}}\, ,\, \textcircled{\scriptsize 1}_{\mathbf{0}}\, ,\, \textcircled{\scriptsize 1}_{\mathbf{0}} \;\overset{(17)}{\Rightarrow}\; \textcircled{\scriptsize 1}_{\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}}}\, ,\, \textcircled{\scriptsize 1}_{-\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}}}$. Likewise $\mathcal{B}$ob's physical model for absorption ${W_{\mathrm{cal}}}^{(\mathcal{B})}$ solely consists of congruent unit actions $w_{\mathbf{1}^{(\mathcal{B})}} : \mathbf{1}^{(\mathcal{B})}_E\big|_{\mathbf{0}}\, ,\, \textcircled{\scriptsize 1}_{\mathbf{0}}\, ,\, \textcircled{\scriptsize 1}_{\mathbf{0}} \;\Rightarrow\; \textcircled{\scriptsize 1}_{\mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}}\, ,\, \textcircled{\scriptsize 1}_{-\mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}}$. Let the velocity of $\mathcal{A}$lice impulse units $\textcircled{\scriptsize 1}_{\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}}}$ be a multiple $\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}} = k \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}$ $k \in \mathbb{N}$ of the unit velocity for $\mathcal{B}$ob $\mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}$. Then $\mathcal{B}$ob's calorimeter measurement ${W_{\mathrm{cal}}}^{(\mathcal{B})}$ is a *refinement* of the absorption measurement by $\mathcal{A}$lice ${W_{\mathrm{cal}}}^{(\mathcal{A})}$.

Lemma 3 *In the* <u>*refinement*</u> *of $\mathcal{A}$lice calorimeter measurements $\mathcal{B}$ob uses equivalent unit objects* $\textcircled{\scriptsize 1}$ *with smaller unit velocity* $\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}} = k \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}$ $k \in \mathbb{N}$. *His calorimeter-collision-cascade* ${W_{\mathrm{cal}}}^{(\mathcal{B})}$ *absorbs $\mathcal{A}$lice dynamical units*

$$\mathrm{RB}^{(\mathcal{B})}\left[\mathbf{1}^{(\mathcal{A})}_{\mathbf{p}} := \textcircled{\scriptsize 1}_{\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}}} \;\Rightarrow\; \textcircled{\scriptsize 1}_{\mathbf{0}}\right] \;\overset{(41)}{=}\; \left(\frac{1}{2}\cdot k^2 - \frac{1}{2}\cdot k\right)\cdot \mathbf{1}^{(\mathcal{B})}_E\big|_{\mathbf{0}} \;\; , \;\; k \cdot \mathbf{1}^{(\mathcal{B})}_{\mathbf{p}} \tag{47}$$

$$\mathrm{RB}^{(\mathcal{B})}\left[\mathbf{1}^{(\mathcal{A})}_E\big|_{\mathbf{0}} \Rightarrow \emptyset\right] \;=\; k^2 \cdot \mathbf{1}^{(\mathcal{B})}_E\big|_{\mathbf{0}} \tag{48}$$

(in a reversible way) in return for dynamical units of $\mathcal{B}$ob.

Proof:

$$\begin{aligned}
\mathrm{RB}^{(\mathcal{B})}\left[\mathbf{1}^{(\mathcal{A})}_E\big|_{\mathbf{0}} \Rightarrow \emptyset\right] &\overset{(17)}{=} \mathrm{RB}^{(\mathcal{B})}\left[\textcircled{\scriptsize 1}_{-\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}}}\, ,\, \textcircled{\scriptsize 1}_{\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}}} \;\Rightarrow\; \textcircled{\scriptsize 1}_{\mathbf{0}}\, ,\, \textcircled{\scriptsize 1}_{\mathbf{0}}\right] \\
&\overset{(46)}{=} \mathrm{RB}^{(\mathcal{B})}\left[\textcircled{\scriptsize 1}_{-k\cdot\mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}} \Rightarrow \textcircled{\scriptsize 1}_{\mathbf{0}}\right] + \mathrm{RB}^{(\mathcal{B})}\left[\textcircled{\scriptsize 1}_{k\cdot\mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}} \Rightarrow \textcircled{\scriptsize 1}_{\mathbf{0}}\right] \\
&\overset{(41)}{=} \left(k^2 - k\right)\cdot \mathbf{1}^{(\mathcal{B})}_E\big|_{\mathbf{0}}\, ,\; k\cdot \mathbf{1}^{(\mathcal{B})}_{-\mathbf{p}}\, ,\; k \cdot \mathbf{1}^{(\mathcal{B})}_{\mathbf{p}} \quad \overset{(17)}{=} \quad k^2 \cdot \mathbf{1}^{(\mathcal{B})}_E\big|_{\mathbf{0}}
\end{aligned}$$

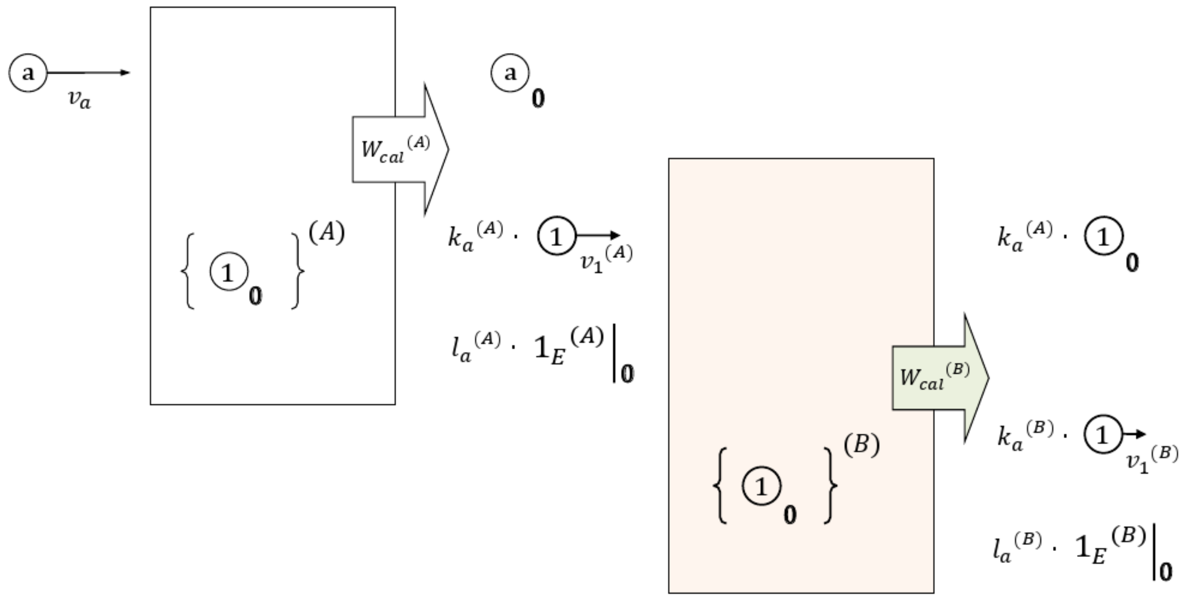

Figure 14: (elastically extracted) dynamical units $\mathbf{1}_E^{(\mathcal{A})}$, $\mathbf{1}_{\mathbf{p}}^{(\mathcal{A})}$ from calorimeter measurement ${W_{\text{cal}}}^{(\mathcal{A})}$ are successively absorbed in - high resolution - calorimeter measurement ${W_{\text{cal}}}^{(\mathcal{B})}$

□

Corollary 1 *The concatenation of $\mathcal{B}$ob's refined calorimeter measurement ${W_{\text{cal}}}^{(\mathcal{A})} * {W_{\text{cal}}}^{(\mathcal{B})}$ in the measurement output of $\mathcal{A}$lice is* <u>*transitive*</u>*. Their reservoir balances - for the absorption of the same particle* $ⓐ$ *- are physically equivalent*

$$\text{RB}^{(\mathcal{A})}\left[ⓐ_{\mathbf{v}_a} \Rightarrow ⓐ_{\mathbf{0}}\right] \quad \sim_{E,\mathbf{p}} \quad \text{RB}^{(\mathcal{B})}\left[ⓐ_{\mathbf{v}_a} \Rightarrow ⓐ_{\mathbf{0}}\right] \quad .$$

Proof: Let incident object $ⓐ_{\mathbf{v}_a}$ have velocity $\mathbf{v}_a = n \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{A})}} = n \cdot (k \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{B})}})$ with $n, k \in \mathbb{N}$. The reservoir balance of absorbing object $ⓐ$ in $\mathcal{A}$lice calorimeter

$$\text{RB}^{(\mathcal{A})}\left[ⓐ_{n\cdot\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}}} \Rightarrow ⓐ_{\mathbf{0}}\right] \overset{(41)}{=} \left(\frac{1}{2}\cdot n^2 - \frac{1}{2}\cdot n\right)\cdot \mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}} \ , \ \ n \cdot \mathbf{1}_{\mathbf{p}}^{(\mathcal{A})}$$

is refined by $\mathcal{B}$ob's successive high resolution calorimeter measurement (see figure 14)

$$\begin{aligned}
&\text{RB}^{(\mathcal{B})}\left[\left(\frac{1}{2}\cdot n^2 - \frac{1}{2}\cdot n\right)\cdot \mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}} \ , \ \ n\cdot \mathbf{1}_{\mathbf{p}}^{(\mathcal{A})}\right] \\
&\overset{(46)}{=} \left(\frac{1}{2}\cdot n^2 - \frac{1}{2}\cdot n\right)\cdot \text{RB}^{(\mathcal{B})}\left[\mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}} \Rightarrow \emptyset\right] + n\cdot \text{RB}^{(\mathcal{B})}\left[\mathbf{1}_{\mathbf{p}}^{(\mathcal{A})} \Rightarrow ①_{\mathbf{0}}\right] \\
&\overset{(48)(47)}{=} \left(\frac{1}{2}\cdot (n\cdot k)^2 - \frac{1}{2}\cdot n\cdot k\right)\cdot \mathbf{1}_E^{(\mathcal{B})}\big|_{\mathbf{0}} , \ n\cdot k\cdot \mathbf{1}_{\mathbf{p}}^{(\mathcal{B})} \overset{(41)}{=} \text{RB}^{(\mathcal{B})}\left[ⓐ_{n\cdot k\cdot\mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}} \Rightarrow ⓐ_{\mathbf{0}}\right] .
\end{aligned}$$

Both calorimeter extracts have same momentum as incident object $ⓐ_{\mathbf{v}_a}$ (see Lemma 6).

□

Lemma 4 *Let $\mathcal{A}$lice move relative to $\mathcal{B}$ob with constant velocity $\mathbf{v}_{\mathcal{A}} = n \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}$ for $n \in \mathbb{N}$. $\mathcal{B}$ob can reproduce the effect of $\mathcal{A}$lice boosted units of energy and momentum*

$$\mathrm{RB}^{(\mathcal{B})}\left[\mathbf{1}_E^{(\mathcal{A})}\big|_{n\cdot\mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}} \Rightarrow \emptyset\right] \;=\; \mathbf{1}_E^{(\mathcal{B})}\big|_{\mathbf{0}} \tag{49}$$

$$\mathrm{RB}^{(\mathcal{B})}\left[\mathbf{1}_{\mathbf{p}}^{(\mathcal{A})} := ①_{\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}}} \;\Rightarrow\; ①_{\mathbf{0}^{(\mathcal{A})}}\right] \;=\; n\cdot\mathbf{1}_E^{(\mathcal{B})}\big|_{\mathbf{0}}\;,\;\; 1\cdot\mathbf{1}_{\mathbf{p}}^{(\mathcal{B})} \tag{50}$$

by expending resting energy units $\mathbf{1}_E^{(\mathcal{B})}\big|_{\mathbf{0}}$ and momentum units $\mathbf{1}_{\mathbf{p}}^{(\mathcal{B})}$ from his own calorimeter reservoir.

Proof: $\mathcal{A}$lice moves relative to $\mathcal{B}$ob with constant velocity $\mathbf{v}_{\mathcal{A}} = \mathbf{v}_{\mathcal{A}}^{(\mathcal{B})} \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}$ with measured value $\mathbf{v}_{\mathcal{A}}^{(\mathcal{B})} = n \in \mathbb{N}$. Measured values of motion for $\mathcal{A}$lice momentum units $①_{\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}}}$, energy units $\mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}^{(\mathcal{A})}}$ and reservoir elements $①_{\mathbf{0}^{(\mathcal{A})}}$ transform - Galilei covariant - by vectorial addition $\mathbf{v}_i^{(\mathcal{B})} = \mathbf{v}_i^{(\mathcal{A})} + \mathbf{v}_{\mathcal{A}}^{(\mathcal{B})}$ (see remark 8). For $\mathcal{B}$ob her momentum unit has velocity

$$\pm\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}} \;=\; \pm 1\cdot\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}} = (n\pm 1)\mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}$$

and her energy unit and reservoir elements are in the state of motion

$$\mathbf{0}^{(\mathcal{A})} \;=\; 0\cdot\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}} = n\cdot\mathbf{v}_{\mathbf{1}^{(\mathcal{B})}} \;\; .$$

$\mathcal{B}$ob can generate the effect of a boosted unit of energy by means of congruent dynamical units from his own (resting) calorimeter reservoir

$$\begin{aligned}
\mathrm{RB}^{(\mathcal{B})}\left[\mathbf{1}_E^{(\mathcal{A})}\big|_{n\cdot\mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}}\right] \;&:=\; \mathrm{RB}^{(\mathcal{B})}\left[\mathbf{1}_E^{(\mathcal{A})}\big|_{n\cdot\mathbf{v_1}} \Rightarrow \emptyset\right] \\
&\overset{(46)}{=}\; \mathrm{RB}^{(\mathcal{B})}\left[\mathbf{1}_E^{(\mathcal{A})}\big|_{n\cdot\mathbf{v_1}}\,,\, ①_{n\cdot\mathbf{v_1}}\,,\, ①_{n\cdot\mathbf{v_1}}\right] \;-\; \mathrm{RB}^{(\mathcal{B})}\left[①_{n\cdot\mathbf{v_1}}\,,\, ①_{n\cdot\mathbf{v_1}}\right] \\
&\overset{(17)}{=}\; \mathrm{RB}^{(\mathcal{B})}\left[①_{(n+1)\cdot\mathbf{v_1}}\,,\, ①_{(n-1)\cdot\mathbf{v_1}}\right] \;-\; \mathrm{RB}^{(\mathcal{B})}\left[①_{n\cdot\mathbf{v_1}}\,,\, ①_{n\cdot\mathbf{v_1}}\right] \\
&\overset{(41)}{=}\; \frac{1}{2}\left[(n+1)^2-(n+1)+(n-1)^2-(n-1)\right]\cdot\mathbf{1}_E^{(\mathcal{B})}\big|_{\mathbf{0}}\;,\;\; 2\cdot n\cdot\mathbf{1}_p^{(\mathcal{B})}\,, \\
&\qquad\qquad -\,(n^2-n)\cdot\mathbf{1}_E^{(\mathcal{B})}\big|_{\mathbf{0}}\;,\;\; -2\cdot n\cdot\mathbf{1}_p^{(\mathcal{B})} \\
&=\; 1\cdot\mathbf{1}_E^{(\mathcal{B})}\big|_{\mathbf{0}}\;,\;\; 0\cdot\mathbf{1}_p^{(\mathcal{B})}
\end{aligned}$$

The reservoir balance specifies the construction of a physical model by $\mathcal{B}$ob. By reversing the process in his calorimeter-collision-cascade (42)

$$W_{\mathrm{cal}}^{(\mathcal{B})^{-1}}:\;\; 1\cdot\mathbf{1}_E^{(\mathcal{B})}\big|_{\mathbf{0}} \;\Rightarrow\; \mathbf{1}_E^{(\mathcal{A})}\big|_{n\cdot\mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}}$$

$\mathcal{B}$ob has to expend one resting energy unit $\mathbf{1}_E^{(\mathcal{B})}\big|_{\mathbf{0}}$ from his calorimeter reservoir $\{①_{\mathbf{0}}\}$ in an organized way to *boost* one resting unit of energy $\mathbf{1}_E^{(\mathcal{A})}\big|_{n\cdot\mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}}$ into state of motion $n\cdot\mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}$.

This - practically instantaneous and reversible - replacement process $W^{(\mathcal{B})^{-1}}_{\text{cal}}$ solely consists of congruent unit actions $w_{\mathbf{1}^{(\mathcal{B})}}: \ \mathbf{1}^{(\mathcal{B})}_{E}\big|_{\mathbf{0}}, \textcircled{1}_{\mathbf{0}}, \textcircled{1}_{\mathbf{0}} \Rightarrow \textcircled{1}_{\mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}}, \textcircled{1}_{-\mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}}$.

Similarly $\mathcal{B}$ob extracts for the absorption of $\mathcal{A}$lice (boosted) unit of momentum transfer $\mathbf{1}^{(\mathcal{A})}_{\mathbf{p}} := \textcircled{1}_{\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}}} \Rightarrow \textcircled{1}_{\mathbf{0}^{(\mathcal{A})}}$ the corresponding calorimeter reservoir balance for deceleration

$$
\begin{aligned}
\mathrm{RB}^{(\mathcal{B})}\left[\mathbf{1}^{(\mathcal{A})}_{\mathbf{p}}\big|_{n\cdot\mathbf{v_1}}\right] &:= \mathrm{RB}^{(\mathcal{B})}\left[\mathbf{1}^{(\mathcal{A})}_{\mathbf{p}}\big|_{\mathbf{0}^{(\mathcal{A})}} \Rightarrow \textcircled{1}_{\mathbf{0}^{(\mathcal{A})}}\right] \\
&= \mathrm{RB}^{(\mathcal{B})}\left[\textcircled{1}_{(n+1)\cdot\mathbf{v_1}} \Rightarrow \textcircled{1}_{n\cdot\mathbf{v_1}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(46)}{=} \mathrm{RB}^{(\mathcal{B})}\left[\textcircled{1}_{(n+1)\cdot\mathbf{v_1}}\right] - \mathrm{RB}^{(\mathcal{B})}\left[\textcircled{1}_{n\cdot\mathbf{v_1}}\right] \\
&\overset{(41)}{=} \frac{1}{2}\left[(n+1)^2-(n+1)\right]\cdot \mathbf{1}^{(\mathcal{B})}_{E}\big|_{\mathbf{0}},\ (n+1)\cdot\mathbf{1}^{(\mathcal{B})}_{\mathbf{p}},\ -\frac{1}{2}(n^2-n)\cdot\mathbf{1}^{(\mathcal{B})}_{E}\big|_{\mathbf{0}},\ -n\cdot\mathbf{1}^{(\mathcal{B})}_{\mathbf{p}} \\
&= n\cdot\mathbf{1}^{(\mathcal{B})}_{E}\big|_{\mathbf{0}},\ 1\cdot\mathbf{1}^{(\mathcal{B})}_{\mathbf{p}}
\end{aligned}
$$

$\mathcal{B}$ob can also reproduce the effect of $\mathcal{A}$lice impulse unit directly from his own calorimeter reservoir by expending $\mathbf{1}^{(\mathcal{B})}_{\mathbf{p}} := \textcircled{1}_{1\cdot\mathbf{v_1}}$ and a domino series of successively boosted energies $\mathbf{1}^{(\mathcal{B})}_{E}\big|_{1\cdot\mathbf{v_1}}, \mathbf{1}^{(\mathcal{B})}_{E}\big|_{2\cdot\mathbf{v_1}} \dots \mathbf{1}^{(\mathcal{B})}_{E}\big|_{n\cdot\mathbf{v_1}}$ in accordance with (49). $\mathcal{B}$ob can produce equivalent physical models resp. replacement processes in many ways.

□

The *physical meaning* of dynamical units $\mathbf{1}_E\big|_{\mathbf{0}}$ and $\mathbf{1}_{\mathbf{p}}$ is determined by pre-theoretical ordering relations $\sim_{E,\mathbf{p}}$ with regard to 'potential to cause action' and 'impetus' {2.3}.

Proposition 6 *The measurement unit of energy $\mathbf{1}_E\big|_{\mathbf{0}}$ represents unit energy $E_{\mathbf{1}}$ and has no momentum.*

$$
\begin{array}{lll}
E\left[\mathbf{1}_E\big|_{\mathbf{0}}\right] =: E_{\mathbf{1}} & \qquad & E\left[\mathbf{1}_{\mathbf{p}}\right] = \frac{1}{2}\cdot E_{\mathbf{1}} \\
\mathbf{p}\left[\mathbf{1}_E\big|_{\mathbf{0}}\right] = 0 & & \mathbf{p}\left[\mathbf{1}_{\mathbf{p}}\right] =: \mathbf{p_1}
\end{array}
\tag{51}
$$

The measurement unit for impulse $\mathbf{1}_{\mathbf{p}} := \textcircled{1}_{\mathbf{v}_1}$ represents unit momentum $\mathbf{p_1}$ and also has energy $\frac{1}{2}\cdot E_{\mathbf{1}}$.

Proof: Energy and impulse are distinguishable aspects but inseparably unified in unit action

$$
w_{\mathbf{1}}: \ \mathbf{1}_E\big|_{\mathbf{0}}, \textcircled{1}_{\mathbf{0}}, \textcircled{1}_{\mathbf{0}} \quad \Rightarrow \quad \textcircled{1}_{-\mathbf{v_1}}, \textcircled{1}_{\mathbf{v_1}} \ .
$$

The energetic unit $\mathbf{1}_E\big|_{\mathbf{0}}$ (i.e. our standardized source of energy) represents the unit energy $E_{\mathbf{1}} := E\left[\mathbf{1}_E\big|_{\mathbf{0}}\right]$. The impulse unit $\mathbf{1}_{\mathbf{p}}$ (i.e. the impulse behavior of our standardized object $\textcircled{1}_{\mathbf{v}_1}$ at unit velocity $\mathbf{v}_1$) represents the unit momentum $\mathbf{p_1} := \mathbf{p}\left[\mathbf{1}_{\mathbf{p}}\right]$ (see Definition 5).

Resting unit of energy $\mathbf{1}_E\big|_{\mathbf{0}}$ can not overrun any other moving object $\textcircled{a}\xrightarrow[v_a]{}$ in a direct collision; if at all it will be overrun by other moving objects. Its *impulse behavior* (10) $\mathbf{1}_E\big|_{\mathbf{0}} <_{\mathbf{p}} \textcircled{a}_{\epsilon\cdot\mathbf{v_1}}$ is weaker than of any other moving object $\forall\,\epsilon > 0$. Its abstract momentum is zero $\mathbf{p}\left[\mathbf{1}_E\big|_{\mathbf{0}}\right] = 0$.

The abstract energy of two impulse units $2 \cdot E\,[\mathbf{1_p}] = E\left[\mathbf{1}_E\big|_{\mathbf{0}}\right]$ equals the energy of our standardized source $\mathbf{1}_E\big|_{\mathbf{0}}$ if and only if the potential effects - of absorbing two (congruent) impulse units

$$[\mathbf{1_p} \Rightarrow ①_{\mathbf{0}}]\ ,\ [\mathbf{1_p} \Rightarrow ①_{\mathbf{0}}] \quad \sim_E \quad \left[\mathbf{1}_E\big|_{\mathbf{0}} \Rightarrow \emptyset\right]$$

resp. of absorbing one energy unit - in the same calorimeter $\{①_{\mathbf{0}}\}$ are interchangeable (Equipollence Principle). If both causes of action have same *energetic behavior* (14) then their abstract energy is the same.

Let a refined calorimetric measurement (42) in a reservoir with equivalent objects $\{①_{\mathbf{0}}\}$ be based on arbitrarily fine (congruent) unit actions

$$w_\epsilon :\ \ \mathbf{1}_E^{(\epsilon)}\big|_{\mathbf{0}}\,,\ ①_{\mathbf{0}}\,,\ ①_{\mathbf{0}} \quad \Rightarrow \quad ①_{-\epsilon\cdot\mathbf{v_1}}\,,\ ①_{\epsilon\cdot\mathbf{v_1}} \qquad \epsilon > 0\ .$$

In the calorimeter-collision-cascade $W_{\text{cal}}^{(\epsilon)}$ both impulse units $2 \cdot \mathbf{1_p}$ and the energetic unit $\mathbf{1}_E\big|_{\mathbf{0}}$ are absorbed - in a reversible way - in return for dynamical units $\mathbf{1}_E^{(\epsilon)}\big|_{\mathbf{0}}$ and $\mathbf{1}_{\mathbf{p}}^{(\epsilon)}$ of the corresponding - high resolution - calorimeter reservoir balance

$$\text{RB}^{(\epsilon)}\,[2 \cdot \mathbf{1_p} \Rightarrow 2 \cdot ①_{\mathbf{0}}] \overset{(46)(47)}{=} \left(\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\right) \cdot \mathbf{1}_E^{(\epsilon)}\big|_{\mathbf{0}}\ ,\ \ \frac{2}{\epsilon} \cdot \mathbf{1}_{\mathbf{p}}^{(\epsilon)} \tag{52}$$

$$\text{RB}^{(\epsilon)}\left[\mathbf{1}_E\big|_{\mathbf{0}} \Rightarrow \emptyset\right] \overset{(48)}{=} \frac{1}{\epsilon^2} \cdot \mathbf{1}_E^{(\epsilon)}\big|_{\mathbf{0}} \tag{53}$$

with $\mathbf{v_1} = \frac{1}{\epsilon} \cdot \mathbf{v}_{\mathbf{1}^{(\epsilon)}}$ as the $\frac{1}{\epsilon}$ multiple of refined unit velocity $\mathbf{v}_{\mathbf{1}^{(\epsilon)}} := \epsilon \cdot \mathbf{v_1}$ (see Remark 9). We examine the equipollence of both causes of action in this redistributed form.

The absorption action of $\mathbf{1}_E\big|_{\mathbf{0}}$ and the absorption action of $2 \cdot \mathbf{1_p}$ is physically specified in the - high resolution - calorimeter $W_{\text{cal}}^{(\epsilon)}$. The extracted dynamical units $\mathbf{1}_E^{(\epsilon)}\big|_{\mathbf{0}}$ and $\mathbf{1}_{\mathbf{p}}^{(\epsilon)}$ take over the role of lowest common *physical* denominators of $\mathbf{1}_E\big|_{\mathbf{0}}$ and $\mathbf{1_p}$. Although the latter are inseparable (from their unit action $w_{\mathbf{1}}$) their respective energy $E\left[\mathbf{1}_E\big|_{\mathbf{0}}\right]$ and $E\,[\mathbf{1_p}]$ is reproduced in - high resolution - calorimeter model $W_{\text{cal}}^{(\epsilon)}$ by common (congruent) 'parts' $\mathbf{1}_E^{(\epsilon)}\big|_{\mathbf{0}}$ and $\mathbf{1}_{\mathbf{p}}^{(\epsilon)}$.

We add to both impulse units $2 \cdot \mathbf{1_p}$ even more moving objects: We add $\frac{2}{\epsilon}$ high resolution impulse units $\mathbf{1}_{-\mathbf{p}}^{(\epsilon)} := ①_{-\epsilon\cdot\mathbf{v_1}}$. In the refinement limit $\epsilon \to 0$ the energy of those additional elements disappears to any adjustable precision

$$\begin{aligned} \lim_{\epsilon\to 0}\ E\left[\frac{2}{\epsilon} \cdot \mathbf{1}_{-\mathbf{p}}^{(\epsilon)}\right] \ &< \ \lim_{\epsilon\to 0}\ E\left[\frac{2}{\epsilon} \cdot \mathbf{1}_{-\mathbf{p}}^{(\epsilon)}\,,\ \frac{2}{\epsilon} \cdot \mathbf{1}_{\mathbf{p}}^{(\epsilon)}\right] \\ &\overset{(17)}{=} \lim_{\epsilon\to 0}\ E\left[\frac{2}{\epsilon} \cdot \mathbf{1}_E^{(\epsilon)}\big|_{\mathbf{0}}\right] \cdot \underbrace{\frac{\epsilon}{\epsilon}}_{=1} \ = \ \lim_{\epsilon\to 0}\ E\left[\underbrace{\frac{1}{\epsilon^2} \cdot \mathbf{1}_E^{(\epsilon)}\big|_{\mathbf{0}}}_{\overset{(48)}{\sim_E}\ \mathbf{1}_E|_{\mathbf{0}}} \cdot 2\epsilon\right] \\ &= \ \lim_{\epsilon\to 0}\ 2\epsilon \cdot E_{\mathbf{1}} \ = \ 0 \cdot E_{\mathbf{1}}\ . \end{aligned} \tag{54}$$

The enlarged system of both impulse units $2 \cdot \mathbf{1}_{\mathbf{p}} \,\cup\, \frac{2}{\epsilon} \cdot \mathbf{1}^{(\epsilon)}_{-\mathbf{p}}$ approximates in the - high resolution - calorimeter the same balance of extracted dynamical units

$$\begin{aligned}
\mathrm{RB}^{(\epsilon)}\left[2 \cdot \mathbf{1}_{\mathbf{p}}\,,\ \frac{2}{\epsilon} \cdot \mathbf{1}^{(\epsilon)}_{-\mathbf{p}} \Rightarrow 2 \cdot ⓵_{\mathbf{0}}\,,\ \frac{2}{\epsilon} \cdot ⓵_{\mathbf{0}}\right] &\overset{(52)}{=} \left(\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\right) \cdot \mathbf{1}^{(\epsilon)}_{E}\Big|_{\mathbf{0}}\ ,\ \underbrace{\frac{2}{\epsilon} \cdot \mathbf{1}^{(\epsilon)}_{\mathbf{p}}\ ,\ \frac{2}{\epsilon} \cdot \mathbf{1}^{(\epsilon)}_{-\mathbf{p}}}_{\overset{(17)}{\sim}_E\ \frac{2}{\epsilon}\cdot \mathbf{1}^{(\epsilon)}_{E}\big|_{\mathbf{0}}} \\
&= \left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\right) \cdot \mathbf{1}^{(\epsilon)}_{E}\Big|_{\mathbf{0}} \\
&\overset{(53)}{=} \mathrm{RB}^{(\epsilon)}\left[\mathbf{1}_{E}\big|_{\mathbf{0}} \Rightarrow \emptyset\right] \ +\ \frac{1}{\epsilon} \cdot \mathbf{1}^{(\epsilon)}_{E}\Big|_{\mathbf{0}} \qquad (55)
\end{aligned}$$

like the absorption of energetic unit $\mathbf{1}_{E}\big|_{\mathbf{0}}$ to any adjustable precision. In the refinement limit $\epsilon \to 0$ they are energetically equivalent

$$\lim_{\epsilon \to 0}\ \mathrm{RB}^{(\epsilon)}\left[2 \cdot \mathbf{1}_{\mathbf{p}}\,,\ \frac{2}{\epsilon} \cdot \mathbf{1}^{(\epsilon)}_{-\mathbf{p}} \Rightarrow 2 \cdot ⓵_{\mathbf{0}}\,,\ \frac{2}{\epsilon} \cdot ⓵_{\mathbf{0}}\right] \ \sim_E\ \mathrm{RB}^{(\epsilon)}\left[\mathbf{1}_{E}\big|_{\mathbf{0}} \Rightarrow \emptyset\right]\ .$$

Therefore the absorption energy of one energetic unit

$$\begin{aligned}
E\left[\mathbf{1}_{E}\big|_{\mathbf{0}}\right] &\overset{(\text{Equip.})}{=} E\left[\mathrm{RB}^{(\epsilon)}\left[\mathbf{1}_{E}\big|_{\mathbf{0}} \Rightarrow \emptyset\right]\right] \\
&\overset{(55)}{=} E\left[\mathrm{RB}^{(\epsilon)}\left[2 \cdot \mathbf{1}_{\mathbf{p}}\,,\ \frac{2}{\epsilon} \cdot \mathbf{1}^{(\epsilon)}_{-\mathbf{p}}\right]\right] \ -\ E\left[\frac{1}{\epsilon} \cdot \mathbf{1}^{(\epsilon)}_{E}\Big|_{\mathbf{0}}\right] \\
&= \lim_{\epsilon \to 0} E\left[\mathrm{RB}^{(\epsilon)}\left[2 \cdot \mathbf{1}_{\mathbf{p}}\,,\ \frac{2}{\epsilon} \cdot \mathbf{1}^{(\epsilon)}_{-\mathbf{p}}\right]\right] \ -\ \underbrace{\lim_{\epsilon \to 0} E\left[\frac{1}{\epsilon} \cdot \mathbf{1}^{(\epsilon)}_{E}\Big|_{\mathbf{0}}\right]}_{\overset{(48)}{=} \lim_{\epsilon\to 0} E\left[\epsilon \cdot \mathbf{1}_{E}\big|_{\mathbf{0}}\right] = 0 \cdot E_{\mathbf{1}}} \\
&\overset{(46)}{=} \underbrace{\lim_{\epsilon \to 0} E\left[\mathrm{RB}^{(\epsilon)}\left[2 \cdot \mathbf{1}_{\mathbf{p}}\right]\right]}_{\overset{(\text{Equip.})}{=} 2 \cdot E[\mathbf{1}_{\mathbf{p}}]} \ +\ \underbrace{\lim_{\epsilon \to 0} E\left[\frac{2}{\epsilon} \cdot \mathbf{1}^{(\epsilon)}_{-\mathbf{p}}\right]}_{\overset{(54)}{=} 0 \cdot E_{\mathbf{1}}}
\end{aligned}$$

is twice as big as the absorption energy of one impulse unit $\mathbf{1}_{\mathbf{p}}$. In particular we immediately conclude together with (17) that $E\left[\mathbf{1}_{-\mathbf{p}}\right] = E\left[\mathbf{1}_{\mathbf{p}}\right]$.

□

Remark 10 *Energetic unit* $\mathbf{1}_{E}\big|_{\mathbf{0}}$ *and momentum unit* $\mathbf{1}_{\mathbf{p}}$ *are distinguishable aspects but inseparably unified in unit action* $w_{\mathbf{1}}$*. The absorption action of* $\mathbf{1}_{E}\big|_{\mathbf{0}}$ *and* $\mathbf{1}_{\mathbf{p}}$ *is substituted by congruent 'partial' actions* w_ϵ*.*[19] *The composition of - inseparable - 'partial' actions* w_ϵ *in calorimeter model* $W^{(\epsilon)}_{\text{cal}}$ *reproduces the absorption action with regard to physical ordering relation energy* $\sim_E$*. The attribute 'partial' refers to appearing as one part in the composition of a physical model. 'Part' is an attribute of the behavior of the physicist.*

[19] Dynamical units $\mathbf{1}_{E}\big|_{\mathbf{0}}$ and $\mathbf{1}_{\mathbf{p}}$ are not chopped into pieces which persist for themselves. Actions $w_{\mathbf{1}}$ and w_ϵ are indivisible - one has them in full or not at all!

Remark 11 *For absorbed particle* $\textcircled{a}_{\mathbf{v}_a} \sim_{E,\mathbf{p}} \mathrm{RB}\,[\textcircled{a}_{\mathbf{v}_a} \Rightarrow \textcircled{a}_{\mathbf{0}}]$ *the calorimeter absorption extract has the same energetic resp. impulse behavior (impulse conservation see Lemma 6).*

Corollary 2 *Two objects* $\textcircled{a}_{\mathbf{v}_a}$ *and* $\textcircled{b}_{\mathbf{v}_b}$ *have same kinetic energy and momentum*

$$\textcircled{a}_{\mathbf{v}_a} \quad \sim_{E,\mathbf{p}} \quad \textcircled{b}_{\mathbf{v}_b}$$

if they are interchangeable with regard to the calorimeter reservoir balance of a - reversible - absorption measurement

$$\mathrm{RB}\,[\textcircled{a}_{\mathbf{v}_a} \Rightarrow \textcircled{a}_{\mathbf{0}}] \quad = \quad \mathrm{RB}\,[\textcircled{b}_{\mathbf{v}_b} \Rightarrow \textcircled{b}_{\mathbf{0}}] \quad .$$

Lemma 5 *Let a composite of* m *bound unit objects* $\underbrace{\textcircled{1} * \cdots * \textcircled{1}}_{m\times} \sim_{m^{(\mathrm{inert})}} \textcircled{a}$ *have same inertial behavior as particle* $\textcircled{a}_{\mathbf{v}_a}$. *Then the reservoir balance for absorption of object* $\textcircled{a}_{\mathbf{v}_a}$ *with velocity* $\mathbf{v}_a$

$$\mathrm{RB}\,[\textcircled{a}_{\mathbf{v}_a} \Rightarrow \textcircled{a}_{\mathbf{0}}] \quad = \quad m \cdot \mathrm{RB}\,[\textcircled{1}_{\mathbf{v}_a} \Rightarrow \textcircled{1}_{\mathbf{0}}] \tag{56}$$

is m *times as big as for the absorption of unit object* $\textcircled{1}_{\mathbf{v}_a}$ *with same velocity* $\mathbf{v}_a$.

Proof: Let tightly connected composite $\textcircled{1} * \cdots * \textcircled{1} \sim_{m^{(\mathrm{inert})}} \textcircled{a}$ and incident particle $\textcircled{a}_{\mathbf{v}_a}$ have same inertial behavior (12). Then in an elastic collision against one another with same initial velocity $v'_a := \frac{1}{2} \cdot v_a$

$$w_L : \quad \textcircled{a}_{\mathbf{v}'_a}\,,\ \textcircled{1} * \cdots * \textcircled{1}_{-\mathbf{v}'_a} \quad \Rightarrow \quad \textcircled{a}_{-\mathbf{v}'_a}\,,\ \textcircled{1} * \cdots * \textcircled{1}_{\mathbf{v}'_a}$$

both objects must change their state of motion in the same anti-symmetrical way. The effect of elastic collision w_L is the exact impulse reversion of both objects. By Galilei-covariance (see Remark 8) there exists an elastic collision

$$w_L : \quad \textcircled{a}_{\mathbf{v}_a}\,,\ \textcircled{1} * \cdots * \textcircled{1}_{\mathbf{0}} \quad \Rightarrow \quad \textcircled{a}_{\mathbf{0}}\,,\ \textcircled{1} * \cdots * \textcircled{1}_{\mathbf{v}_a}$$

where incident particle $\textcircled{a}_{\mathbf{v}_a}$ comes to rest in return for composite object getting into (same) motion. It is the same physical process as seen by another observer who moves with relative velocity $-\mathbf{v}'_a$.

We mediate this elastic collision w_L indirectly by our physical model, the calorimeter-collision-cascade W_{cal}. The reservoir balance for this interaction of motion

$$\mathrm{RB}\,[\textcircled{a}_{\mathbf{v}_a}\,,\ \textcircled{1} * \cdots * \textcircled{1}_{\mathbf{0}} \ \Rightarrow\ \textcircled{a}_{\mathbf{0}}\,,\ \textcircled{1} * \cdots * \textcircled{1}_{\mathbf{v}_a}]$$

$$\stackrel{(46)}{=} \quad \underbrace{\mathrm{RB}\,[\textcircled{a}_{\mathbf{v}_a} \Rightarrow \textcircled{a}_{\mathbf{0}}]}_{=:\ k_a \cdot \mathbf{1}_E\big|_{\mathbf{0}},\ l_a \cdot \mathbf{1}_{\mathbf{p}}} \quad - \quad \underbrace{\mathrm{RB}\,[\textcircled{1} * \cdots * \textcircled{1}_{\mathbf{v}_a} \Rightarrow m \cdot \textcircled{1}_{\mathbf{0}}]}_{=:\ k_m \cdot \mathbf{1}_E\big|_{\mathbf{0}},\ l_m \cdot \mathbf{1}_{\mathbf{p}}}$$

$$= \quad (k_a - k_m) \cdot \mathbf{1}_E\big|_{\mathbf{0}} \ + \ (l_a - l_m) \cdot \mathbf{1}_{\mathbf{p}} \ \stackrel{!}{=} \ 0$$

decomposes into a sum of separate reservoir balances for absorbing respective objects individually (see Lemma 2). The total number of extracted dynamical units $\mathbf{1}_E\big|_{\mathbf{0}}$ and $\mathbf{1_p}$ has to vanish because an *elastic* action w_L cannot leave any mark in the (mediating) calorimeter. The total reservoir balance

$$\mathbf{p}\left[(k_a - k_m)\cdot \mathbf{1}_E\big|_{\mathbf{0}}\,,\ (l_a - l_m)\cdot \mathbf{1_p}\right] \overset{(51)}{=} \mathbf{p}\left[(l_a - l_m)\cdot \mathbf{1_p}\right] \overset{!}{=} 0 \tag{57}$$

cannot have momentum. It also cannot have energy

$$E\ [\ (k_a - k_m)\cdot \mathbf{1}_E\big|_{\mathbf{0}}\,,\ \underbrace{(l_a - l_m)\cdot \mathbf{1_p}}_{\overset{(57)}{=}0}\ ] \overset{!}{=} 0\ .$$

Thus in the absorption of both objects $\textcircled{a}_{\mathbf{v}_a}$ resp. $\textcircled{1}*\cdots*\textcircled{1}_{\mathbf{v}_a}$ the number $l_a \overset{!}{=} l_m$ of extracted impulse units $\mathbf{1_p}$ and the number $k_a \overset{!}{=} k_m$ of extracted of energetic units $\mathbf{1}_E\big|_{\mathbf{0}}$ must be the same. Therefore in the reservoir balance for absorption of object $\textcircled{a}_{\mathbf{v}_a}$

$$\begin{aligned}
\mathrm{RB}\left[\textcircled{a}_{\mathbf{v}_a} \Rightarrow \textcircled{a}_{\mathbf{0}}\right] &= k_i \cdot \mathbf{1}_E\big|_{\mathbf{0}}\,,\ l_i\cdot \mathbf{1_p} \qquad\qquad i = a, m\\
&= \mathrm{RB}\left[\textcircled{1}*\cdots*\textcircled{1}_{\mathbf{v}_a} \Rightarrow \textcircled{1}*\cdots*\textcircled{1}_{\mathbf{0}}\right]\\
&\overset{(46)}{=} m\cdot \mathrm{RB}\left[\textcircled{1}_{\mathbf{v}_a} \Rightarrow \textcircled{1}_{\mathbf{0}}\right]
\end{aligned}$$

we extract m times the number of equivalent dynamical units as for the absorption of equally moving unit object $\textcircled{1}_{\mathbf{v}_a}$.

□

Proposition 7 *An incident object* $\textcircled{a}_{\mathbf{v}_a}$ *with inertial mass* $m_a = m_a^{(\mathcal{A})}\cdot m_{\mathbf{1}^{(\mathcal{A})}}$ *and velocity* $\mathbf{v}_a = \underbrace{\mathbf{v}_a^{(\mathcal{A})}}_{=:n}\cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{A})}}$ *has* <u>*kinetic*</u> *energy and momentum*

$$\begin{aligned}
E\left[\textcircled{a}_{\mathbf{v}_a}\right] &= \left\{\tfrac{1}{2}\cdot m_a^{(\mathcal{A})}\cdot \mathbf{v}_a^{(\mathcal{A})^2}\right\}\cdot E_{\mathbf{1}^{(\mathcal{A})}}\\
\mathbf{p}\left[\textcircled{a}_{\mathbf{v}_a}\right] &= \left\{m_a^{(\mathcal{A})}\cdot \mathbf{v}_a^{(\mathcal{A})}\right\}\cdot \mathbf{p}_{\mathbf{1}^{(\mathcal{A})}}\ .
\end{aligned} \tag{58}$$

Proof: In $\mathcal{A}$lice calorimeter-collision-cascade W_{cal} incident object $\textcircled{a}_{\mathbf{v}_a}$ is absorbed - in a reversible way - in return for the extraction of equivalent dynamical units $\mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}}$ and $\mathbf{1}_{\mathbf{p}}^{(\mathcal{A})}$ (see Remark 9). By the Equipollence Principle the kinetic energy is conserved

$$\begin{aligned}
E\left[\textcircled{a}_{\mathbf{v}_a}\right] &= E\left[\mathrm{RB}\left[\textcircled{a}_{\mathbf{v}_a} \Rightarrow \textcircled{a}_{\mathbf{0}}\right]\right]\\
&\overset{(56)}{=} E\left[m_a^{(\mathcal{A})}\cdot \mathrm{RB}\left[\textcircled{1}_{\mathbf{v}_a} \Rightarrow \textcircled{1}_{\mathbf{0}}\right]\right]\\
&\overset{(41)}{=} m_a^{(\mathcal{A})}\cdot\left\{\left(\frac{1}{2}\cdot n^2 - \frac{1}{2}\cdot n\right)\cdot E\left[\mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}}\right] + n\cdot E\left[\mathbf{1}_{\mathbf{p}}^{(\mathcal{A})}\right]\right\}\\
&\overset{(51)}{=} m_a^{(\mathcal{A})}\cdot\frac{1}{2}\cdot n^2\cdot E_{\mathbf{1}^{(\mathcal{A})}} = \left\{\frac{1}{2}\cdot m_a^{(\mathcal{A})}\cdot \mathbf{v}_a^{(\mathcal{A})^2}\right\}\cdot E_{\mathbf{1}^{(\mathcal{A})}}
\end{aligned}$$

and reproduced by $\left\{\frac{1}{2}\cdot m_a^{(\mathcal{A})}\cdot \mathbf{v}_a^{(\mathcal{A})^2}\right\}$ equivalent energetic units $\mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}}$ from $\mathcal{A}$lice reservoir. Calorimeter extract has same impulse as incident object $\textcircled{a}_{\mathbf{v}_a}$ (see Lemma 6). Its impulse

$$\begin{aligned}\mathbf{p}\left[\textcircled{a}_{\mathbf{v}_a}\right] &= \mathbf{p}\left[\mathrm{RB}\left[\textcircled{a}_{\mathbf{v}_a}\Rightarrow \textcircled{a}_{\mathbf{0}}\right]\right]\\ &\overset{(41)(51)}{=} m_a^{(\mathcal{A})}\cdot n\cdot \mathbf{p}_{\mathbf{1}^{(\mathcal{A})}} = \left\{m_a^{(\mathcal{A})}\cdot \mathbf{v}_a^{(\mathcal{A})}\right\}\cdot \mathbf{p}_{\mathbf{1}^{(\mathcal{A})}}\end{aligned}$$

is reproduced by $\left\{m_a^{(\mathcal{A})}\mathbf{v}_a^{(\mathcal{A})}\right\}$ congruent impulse units $\mathbf{1}_{\mathbf{p}}^{(\mathcal{A})}$ from $\mathcal{A}$lice reservoir.

□

Theorem 2 *Let $\mathcal{A}$lice provide a reservoir with equivalent objects $\{\textcircled{1}_{\mathbf{0}}\}$ and congruent unit actions $w_{\mathbf{1}}$. The calorimeter-collision-cascade W_{cal} is a physical model for the absorption action of incident object $\textcircled{a}_{\mathbf{v}_a}$ with velocity $\mathbf{v}_a$ in her calorimeter $\{\textcircled{1}_{\mathbf{0}}\}$*

$$W_{\text{cal}}:\quad \textcircled{a}_{\mathbf{v}_a},\ \{\textcircled{1}_{\mathbf{0}}\}\quad\Rightarrow\quad \textcircled{a}_{\mathbf{0}},\ k_a^{(\mathcal{A})}\cdot \mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}},\ l_a^{(\mathcal{A})}\cdot \mathbf{1}_{\mathbf{p}}^{(\mathcal{A})}\ .$$

Incident object $\textcircled{a}_{\mathbf{v}_a}$ comes to rest in return for generating - in an organized and reversible way - a certain number of equivalent dynamical units from the calorimeter reservoir $\{\textcircled{1}_{\mathbf{0}}\}$. In her physical model $W_{\text{cal}}^{(\mathcal{A})}$ $\mathcal{A}$lice can count the number of extracted

- $\sharp\left\{\mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}}\right\} =: k_a^{(\mathcal{A})}$ *equivalent energetic units $\mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}}$ (representing unit energy $E_{\mathbf{1}^{(\mathcal{A})}}$)*
- $\sharp\left\{\mathbf{1}_{\mathbf{p}}^{(\mathcal{A})}\right\} =: l_a^{(\mathcal{A})}$ *equivalent impulse units $\mathbf{1}_{\mathbf{p}}^{(\mathcal{A})} := \textcircled{1}_{\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}}}$ (representing unit momentum $\mathbf{p}_{\mathbf{1}^{(\mathcal{A})}}$ and energy $\frac{1}{2}\cdot E_{\mathbf{1}^{(\mathcal{A})}}$)*
- $\sharp\{\textcircled{1}\} =: m_a^{(\mathcal{A})}$ *amount of matter in a composite of unit objects $\underbrace{\textcircled{1}*\cdots*\textcircled{1}}_{m\times} \sim_{m^{(\text{inert})}} \textcircled{a}$ with same inertia as incident object $\textcircled{a}_{\mathbf{v}_a}$ and kinematically*
- $\sharp\{\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}}\} =: \mathbf{v}_a^{(\mathcal{A})}$ *multiplicity of unit velocity $\mathbf{v}_a = \mathbf{v}_a^{(\mathcal{A})}\cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{A})}}$.*

In physical model W_{cal} kinetic energy and momentum of incident object $\textcircled{a}_{\mathbf{v}_a}$ is conserved and redistributed onto equivalent dynamical units $\mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}}$ and $\mathbf{1}_{\mathbf{p}}^{(\mathcal{A})}$. By the congruence principle the kinetic energy and momentum of incident particle $\textcircled{a}_{\mathbf{v}_a}$

$$\begin{aligned}E\left[\textcircled{a}_{\mathbf{v}_a}\right] &=: E_a^{(\mathcal{A})}\cdot E_{\mathbf{1}^{(\mathcal{A})}} = \left\{\tfrac{1}{2}\cdot m_a^{(\mathcal{A})}\cdot \mathbf{v}_a^{(\mathcal{A})^2}\right\}\cdot E_{\mathbf{1}^{(\mathcal{A})}}\\ \mathbf{p}\left[\textcircled{a}_{\mathbf{v}_a}\right] &=: \mathbf{p}_a^{(\mathcal{A})}\cdot \mathbf{p}_{\mathbf{1}^{(\mathcal{A})}} = \left\{m_a^{(\mathcal{A})}\cdot \mathbf{v}_a^{(\mathcal{A})}\right\}\cdot \mathbf{p}_{\mathbf{1}^{(\mathcal{A})}}\ .\end{aligned}\tag{59}$$

<u>become measurable</u> as a multiple of $\mathcal{A}$lice unit energy $E_{\mathbf{1}^{(\mathcal{A})}}$ and unit momentum $\mathbf{p}_{\mathbf{1}^{(\mathcal{A})}}$ - with equations (59) obtained for the construction of physical model W_{cal} in Galilei-Kinematics.

3.1.6 Interrelation

The calorimeter-collision-cascade constitutes a relation between independently defined basic dynamical measures

- inertial mass $m\,[\textcircled{\scriptsize a}] =: m_a^{(\mathcal{A})} \cdot m_{\mathbf{1}^{(\mathcal{A})}}$
- kinetic energy $E\,[\textcircled{\scriptsize a}_{\mathbf{v}_a}] =: E_a^{(\mathcal{A})} \cdot E_{\mathbf{1}^{(\mathcal{A})}}$
- momentum $\mathbf{p}\,[\textcircled{\scriptsize a}_{\mathbf{v}_a}] =: \mathbf{p}_a^{(\mathcal{A})} \cdot \mathbf{p}_{\mathbf{1}^{(\mathcal{A})}}$.

Each physical measure is quantified by the number $m_a^{(\mathcal{A})}, E_a^{(\mathcal{A})}$ resp. $\mathbf{p}_a^{(\mathcal{A})}$ of equivalent elements in the physical model $\textcircled{\scriptsize 1}, \mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}}$ resp. $\textcircled{\scriptsize 1}_{\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}}}$ and the unit measure which each of them represents $m_{\mathbf{1}^{(\mathcal{A})}}, E_{\mathbf{1}^{(\mathcal{A})}}$ resp. $\mathbf{p}_{\mathbf{1}^{(\mathcal{A})}}$. The relation between $\mathcal{A}$lice - independently - measurable values $m_a^{(\mathcal{A})}, E_a^{(\mathcal{A})}$ and $\mathbf{p}_a^{(\mathcal{A})}$ follows from the interrelation of respective unit objects $\textcircled{\scriptsize 1}$, energetic units $\mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}}$ and impulse units $\textcircled{\scriptsize 1}_{\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}}}$ in the physical model W_{cal}. We <u>justify</u> equations between physical quantities of energy and momentum as a genetic consequence of underlying measurement operations - of controlling the coupling in a layout of solely congruent unit actions $w_{\mathbf{1}}$ for physical model $W_{\text{cal}} := w_{\mathbf{1}} * \ldots * w_{\mathbf{1}}$. Physics is the mother of its Mathematics in empirical practice.

When $\mathcal{A}$lice constructs the calorimeter-collision-cascade in Galilei-Kinematics her measured values (German: Meßwerte) for kinetic energy $E_a^{(\mathcal{A})}$, impulse $\mathbf{p}_a^{(\mathcal{A})}$, inertial mass $m_a^{(\mathcal{A})}$ and velocity $\mathbf{v}_a^{(\mathcal{A})}$ satisfy relations

$$\begin{aligned} E_a^{(\mathcal{A})} &= \frac{1}{2} \cdot m_a^{(\mathcal{A})} \cdot v_a^{(\mathcal{A})\,2} \\ \mathbf{p}_a^{(\mathcal{A})} &= m_a^{(\mathcal{A})} \cdot \mathbf{v}_a^{(\mathcal{A})} \\ E_a^{(\mathcal{A})} &= \frac{p_a^{(\mathcal{A})\,2}}{2 \cdot m_a^{(\mathcal{A})}} \quad . \end{aligned} \tag{60}$$

Wallot [12] calls equations between *physical quantities* like (60) in which numerical values occur in the form

$$E_a^{(\mathcal{A})} := \frac{E\,[\textcircled{\scriptsize a}_{\mathbf{v}_a}]}{E_{\mathbf{1}^{(\mathcal{A})}}} \qquad p_a^{(\mathcal{A})} := \frac{p\,[\textcircled{\scriptsize a}_{\mathbf{v}_a}]}{p_{\mathbf{1}^{(\mathcal{A})}}} \qquad m_a^{(\mathcal{A})} := \frac{m\,[\textcircled{\scriptsize a}]}{m_{\mathbf{1}^{(\mathcal{A})}}} \qquad v_a^{(\mathcal{A})} := \frac{v_a}{v_{\mathbf{1}^{(\mathcal{A})}}}$$

measure/unit measure - 'tailored quantitative equations' (German: zugeschnittene Größengleichungen). Those quantitative equations are tailored for particular measurement units (here: for basic units of energy, impulse and inertia).

The measurement theoretical foundation of basic dynamical quantities is circularity free. We do not presuppose equations of motion and other mathematical relations between basic dynamical quantities (e.g. mathematical formulations of conservation laws, symmetries etc.). Every new basic measure: energy, momentum, inertial mass has to be explained in words or by examples because definition-*equations* for basic quantities do not exist [12].

With Ruben [15] we acknowledge the important methodical distinction between measurement object and measurement unit. Both measurement object and measurement unit are physical objects. In the act of a measurement - which is always a *pair comparison* between measurement object and material model (see Remark 4) - they have different functions. Physicists specify the measurement object (with regard to its energetic and impulse behavior) while they have to provide a measurement unit in a suitable way [22]. Measurement units can be refined i.e. substituted by finer units (e.g. unit action $w_{\mathbf{1}}$ is substituted by refined unit action w_ϵ in the impulse reversion process (30) or in the calorimeter-collision-cascade (42)) but they are never chopped into pieces (see Remark 10). We have the energetic and impulse behavior of unit action $w_{\mathbf{1}}$ in full or nothing at all.

In the basic measurement we construct a physical model which solely consists of congruent unit actions $w_{\mathbf{1}}$. Taken by themselves - as inseparable measurement *unit* - they are also *unquantified* but these units are *congruent* among one another (similarly to light-clocks in basic measurements of relativistic Kinematics - see Remark 3). By coupling dynamical units from an external reservoir $\left\{\mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}}, ⓵_{\mathbf{0}}\right\}$ we assemble a sequence of unit actions $w_{\mathbf{1}}*\ldots*w_{\mathbf{1}}$ by physical operations: They are respectively associated in the standardized object ⓵ which in between each action moves freely. The course of their couplings is controlled from the outside by (a collective of) physicists. By means of this model the measurement *object* is *metricized* - in an observer independent reproducible way! The physical ordering relation with regard to energy and momentum {2.3} in other actions e.g. elastic collision {3.1.4}, absorption action {3.1.5} and impulse composition {3.2} becomes measurable. The congruence principle is constitutive for basic measurements.

3.2 Momentum

According to the Principle of Inertia moving bodies move on their own. Their state of motion is preserved unless they are effected by an external cause [2]. In a deceleration the moving body acts against the external object - colloquially spoken - with 'striking power' or 'impetus' or 'impulse' into the direction of its motion. We can compare the impulse behavior of two moving objects $ⓐ_{\mathbf{v}_a}$, $ⓑ_{\mathbf{v}_b}$ directly - i.e. without further differentiation of spatiotemporal details of the interaction - in a physical ordering relation (see Definition 2). The ordering relation with regard to momentum is determined by the physical behavior in a collision: Object $ⓐ_{\mathbf{v}_a}$ acts with *same impulse* against object $ⓑ_{\mathbf{v}_b}$

$$ⓐ_{v_a^{(\mathcal{A})}\cdot\mathbf{v_1}} \sim_{\mathbf{p}} ⓑ_{-v_b^{(\mathcal{A})}\cdot\mathbf{v_1}}$$

if in an - against one another directed - inelastic collision none of the two bodies overruns the other

$$ⓐ_{v_a^{(\mathcal{A})}\cdot\mathbf{v_1}},\ ⓑ_{-v_b^{(\mathcal{A})}\cdot\mathbf{v_1}} \Rightarrow ⓐ*ⓑ_{\mathbf{0}} .$$

The (joint) collision product $ⓐ*ⓑ_{\mathbf{0}}$ moves neither into the former direction of object $ⓐ_{\mathbf{v}_a}$ to the right nor into the former direction of object $ⓑ_{\mathbf{v}_b}$ to the left. If the (joint) product of the collision continues moving into the direction of $\textcircled{a}_{\mathbf{v}_a}$ then the latter has more momentum

$$\textcircled{a}_{\mathbf{v}_a} \;\; >_{\mathbf{p}} \;\; \textcircled{b}_{\mathbf{v}_b}$$

than object $\textcircled{b}_{\mathbf{v}_b}$. Vice versa if $\textcircled{b}_{\mathbf{v}_b}$ overruns its collision partner $\textcircled{a}_{\mathbf{v}_a}$ then the latter has less striking power $\textcircled{a}_{\mathbf{v}_a} <_{\mathbf{p}} \textcircled{b}_{\mathbf{v}_b}$.

We associate the momentum of two moving objects $\textcircled{a}_{\mathbf{v}_a}$ and $\textcircled{b}_{\mathbf{v}_b}$ by coupling two consecutive absorption actions: We expend the impulse of both moving objects against the same external (system of) object $\textcircled{1}_{\mathbf{v}}$

$$\textcircled{a}_{\mathbf{v}_a}\,,\; \textcircled{b}_{\mathbf{v}_b}\,,\; \textcircled{1}_{\mathbf{v}} \;\;\Rightarrow\;\; \textcircled{a}_{\mathbf{0}}\,,\; \textcircled{b}_{\mathbf{0}}\,,\; \textcircled{1}_{\mathbf{v}'}$$

such that in the final state both objects $\textcircled{a}_{\mathbf{0}}$ and $\textcircled{b}_{\mathbf{0}}$ come to rest. The commutativity in the order of coupling both objects against the same external system $\{\textcircled{1}_{\mathbf{v}}\}$ is guaranteed by our (equivalent) physical model for the absorption action W_{cal}. We represent the impulse unit $\mathbf{1}_{\mathbf{p}}^{(\mathcal{A})}$ by the impulse behavior of $\mathcal{A}$lice standardized objects $\textcircled{1}_{\mathbf{v_1}}$ in the deceleration from the state of unit motion $\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}}$ into the state of rest.

We provide a physical model for the impulse of object $\textcircled{a}_{\mathbf{v}_a}$ from the calorimeter-collision-cascade W_{cal}. Our physical model for absorption action - of incident object $\textcircled{a}_{\mathbf{v}_a} \Rightarrow \textcircled{a}_{\mathbf{0}}$ in a calorimeter reservoir $\{\textcircled{1}_{\mathbf{0}}\}$ - solely consists of congruent unit actions $w_{\mathbf{1}^{(\mathcal{A})}}$. In return for absorbing object $\textcircled{a}_{\mathbf{v}_a}$ the reservoir extracts

$$\text{RB}\,[\textcircled{a}_{\mathbf{v}_a} \Rightarrow \textcircled{a}_{\mathbf{0}}] \;\;=\;\; k_a \cdot \mathbf{1}_E\big|_{\mathbf{0}}\,,\; l_a \cdot \mathbf{1}_{\mathbf{p}}$$

k_a energetic units $\mathbf{1}_E\big|_{\mathbf{0}}$ and l_a momentum units $\mathbf{1}_{\mathbf{p}}$ (see Definition 7). The momentum is carried by part of the calorimeter extract

$$\mathbf{p}\,[\text{RB}\,[\textcircled{a}_{\mathbf{v}_a} \Rightarrow \textcircled{a}_{\mathbf{0}}]] \;\;\stackrel{(51)}{=}\;\; \mathbf{p}\,[\textcircled{1}_{\mathbf{v_1}}, \ldots, \textcircled{1}_{\mathbf{v_1}}] \;\;,$$

by $l_a \stackrel{(41)(56)}{:=} \frac{m_a}{m_{\mathbf{1}}} \cdot \frac{\mathbf{v}_a}{\mathbf{v_1}}$ impulse units $\mathbf{1}_{\mathbf{p}} := \textcircled{1}_{\mathbf{v_1}}$. From the initially resting reservoir $\{\textcircled{1}_{\mathbf{0}}\}$ a swarm $\textcircled{1}_{\mathbf{v_1}}, \ldots, \textcircled{1}_{\mathbf{v_1}} =: l_a \cdot \textcircled{1}_{\mathbf{v_1}}$ of comoving impulse units $\textcircled{1}_{\mathbf{v_1}}$ is extracted with unit velocity $\mathbf{v_1}$ into the direction of motion of absorbed particle $\textcircled{a}_{\mathbf{v}_a}$. That partial output of the reservoir balance is our physical model for momentum. The swarm has same impulse behavior

$$\textcircled{1}_{\mathbf{v_1}}, \ldots, \textcircled{1}_{\mathbf{v_1}} \;\;\sim_{\mathbf{p}}\;\; \textcircled{a}_{\mathbf{v}_a}$$

as incident object $\textcircled{a}_{\mathbf{v}_a}$ (see Lemma 6). In her physical model $\mathcal{A}$lice can count the number of equivalent impulse units $\textcircled{1}_{\mathbf{v_1}}$. By the congruence principle the (directed) impulse of object $\textcircled{a}_{\mathbf{v}_a}$ becomes measurable by the number $\mathbf{p}_a^{(\mathcal{A})}$ of impulse units $\textcircled{1}_{\mathbf{v_1}}$

$$\mathbf{p}\,[\textcircled{a}_{\mathbf{v}_a}] =: \mathbf{p}_a^{(\mathcal{A})} \cdot \mathbf{p}_{\mathbf{1}^{(\mathcal{A})}}$$

and its unit impulse $\mathbf{p}_{\mathbf{1}^{(\mathcal{A})}} := \mathbf{p}\,[\textcircled{1}_{\mathbf{v_1}}]$. This method of metrization is universally *reproducible* and intersubjectively interchangeable with regard to the individual physicist $\mathcal{A}$lice or $\mathcal{B}$ob.

Lemma 6 *In the calorimeter-collision-cascade the (physical) impulse model* $\textcircled{1} * \ldots * \textcircled{1}_{\mathbf{v_1}}$

$$\textcircled{1} * \ldots * \textcircled{1}_{\mathbf{v_1}} \quad \sim_{\mathbf{p}} \quad \textcircled{a}_{\mathbf{v}_a}$$

has the same impulse behavior as incident particle $\textcircled{a}_{\mathbf{v}_a}$. *In physical model for absorption action* W_{cal} *the transferred momentum* $\mathbf{p}\,[\textcircled{a}_{\mathbf{v}_a}] \;=\; \mathbf{p}\,[\mathrm{RB}\,[\textcircled{a}_{\mathbf{v}_a} \Rightarrow \textcircled{a}_{\mathbf{0}}]]$ *is* ***conserved***.

Proof: Incident particle $\textcircled{a}_{\mathbf{v}_a}$ and its (physical) impulse model $\textcircled{1} * \ldots * \textcircled{1}_{\mathbf{v_1}}$ have identical abstract momentum $\mathbf{p}\,[\textcircled{a}_{\mathbf{v}_a}] = \mathbf{p}\,[\textcircled{1} * \ldots * \textcircled{1}_{\mathbf{v_1}}]$ if and only if none of both overruns the other in an against one another directed collision

$$w: \quad \textcircled{a}_{\mathbf{v}_a}\,,\ \textcircled{1} * \ldots * \textcircled{1}_{\mathbf{v_1}} \quad \Rightarrow \quad \textcircled{a}_{\mathbf{0}}\,,\ \textcircled{1} * \ldots * \textcircled{1}_{\mathbf{0}} \quad .$$

In inelastic collision w both $\textcircled{a}_{\mathbf{v}_a}$ - with opposite directed velocity $\mathbf{v}_a := -n \cdot \mathbf{v_1} \quad n \in \mathbb{N}$ - and bound composite $\textcircled{1} * \ldots * \textcircled{1}_{\mathbf{v_1}}$ come to rest (see *physical* Definition 2).[20]

We can mediate this inelastic collision action $w_{(m)}$ by our physical model of - reversible - absorption actions W_{cal} in a calorimeter. The reservoir balance for this interaction of motion

$$\begin{aligned}
&\mathrm{RB}\,[\textcircled{a}_{\mathbf{v}_a}\,,\ \textcircled{1} * \ldots * \textcircled{1}_{\mathbf{v_1}} \Rightarrow \textcircled{a}_{\mathbf{0}}\,,\ \textcircled{1} * \ldots * \textcircled{1}_{\mathbf{0}}] \\
\overset{(46)}{=}\ &\underbrace{\mathrm{RB}\,[\textcircled{a}_{\mathbf{v}_a} \Rightarrow \textcircled{a}_{\mathbf{0}}]}_{\overset{(41)}{=}\ \left(\frac{1}{2}\cdot n^2 - \frac{1}{2}\cdot n\right)\cdot \mathbf{1}_E\big|_{\mathbf{0}}\,,\ n\cdot\mathbf{1}_{-\mathbf{p}}} \quad + \quad \underbrace{\mathrm{RB}\,[\textcircled{1} * \ldots * \textcircled{1}_{\mathbf{v_1}} \Rightarrow \textcircled{1} * \ldots * \textcircled{1}_{\mathbf{0}}]}_{= \mathrm{RB}\left[\textcircled{1}_{\mathbf{v_1}}\,,\ldots,\,\textcircled{1}_{\mathbf{v_1}} \Rightarrow \textcircled{1}_{\mathbf{0}}\,,\ldots,\,\textcircled{1}_{\mathbf{0}}\right] \equiv n\cdot\mathbf{1}_{\mathbf{p}}} \\
\overset{(17)}{=}\ &\left(\tfrac{1}{2}\cdot n^2 + \tfrac{1}{2}\cdot n\right)\cdot \mathbf{1}_E\big|_{\mathbf{0}}
\end{aligned}$$

decomposes into a sum of separate reservoir balances for absorbing incident particle $\textcircled{a}_{\mathbf{v}_a}$ and its impulse model $\textcircled{1} * \ldots * \textcircled{1}_{\mathbf{v_1}}$ (where we assume that its tight linkage can be opened and closed without practical consequences). In *mediated* inelastic collision $w_{(m)}$ both sources of impulse come to rest without overrunning the other

$$w_{(m)}: \quad \textcircled{a}_{\mathbf{v}_a}\,,\ \textcircled{n}_{\mathbf{v_1}} \quad \Rightarrow \quad \textcircled{a}_{\mathbf{0}}\,,\ \textcircled{n}_{\mathbf{0}}\,,\ \left(\frac{1}{2}\cdot n^2 + \frac{1}{2}\cdot n\right)\cdot \mathbf{1}_E\big|_{\mathbf{0}} \tag{61}$$

in return for the extraction of $\frac{1}{2}\cdot n^2 + \frac{1}{2}\cdot n$ energetic units $\mathbf{1}_E\big|_{\mathbf{0}}$.

In *direct* inelastic collision $w_{(d)}$ incident particle $\textcircled{a}_{\mathbf{v}_a}$ and its impulse model $\textcircled{n}_{\mathbf{v_1}}$

$$w_{(d)}: \quad \textcircled{a}_{\mathbf{v}_a}\,,\ \textcircled{n}_{\mathbf{v_1}} \quad \Rightarrow \quad \textcircled{a} * \textcircled{n}_{\mathbf{v}'} \quad . \tag{62}$$

form a bound object $\textcircled{a} * \textcircled{n}_{\mathbf{v}'}$ with velocity $\mathbf{v}'$ and bounding energy E^*. In the case of calorimeter-mediated inelastic collision $w_{(m)}$ that bounding energy E^* of both objects

$$\textcircled{a} * \textcircled{n}_{\mathbf{v}'} \quad \sim_E \quad \textcircled{a}_{\mathbf{0}}\,,\ \textcircled{n}_{\mathbf{0}}\,,\ \left(\frac{1}{2}\cdot n^2 + \frac{1}{2}\cdot n\right)\cdot \mathbf{1}_E\big|_{\mathbf{0}}$$

[20] Equality of momentum $\mathbf{p}\,[\textcircled{a}_{\mathbf{v}_a}] = \mathbf{p}\,[\textcircled{b}_{\mathbf{v}_b}]$ is not determined by attributing the same arbitrary number to objects $\textcircled{a}_{\mathbf{v}_a}$ and $\textcircled{b}_{\mathbf{v}_b}$ (e.g. by means of a simple mathematical function $P:$ 'moving bodies' $\to \mathbb{R}$ so that $P(\textcircled{a}_{\mathbf{v}_a}) = P(\textcircled{a}_{\mathbf{v}_b})$ posses the same formal value). Equivalence of momentum is determined by concrete physical behavior of both objects - namely in the case that they do not overrun one another in an against one another directed collision.

is expended onto $\frac{1}{2}\cdot n^2+\frac{1}{2}\cdot n$ energetic units $\mathbf{1}_E\big|_{\mathbf{0}}$ from the calorimeter reservoir (conservation of energy by Equipollence Principle) and both *unbound* objects $\textcircled{a}_{\mathbf{0}}$, $\textcircled{n}_{\mathbf{0}}$ come to rest. We have to show that - independently how the bounding energy E^* is absorbed - the bound object $\textcircled{a} * \textcircled{n}_{\mathbf{v'}}$ must come to rest as well $\mathbf{v'} \overset{!}{=} \mathbf{0}$.

Like every other moving object the bound system $\textcircled{a}*\textcircled{n}_{\mathbf{v'}}$ can be absorbed in a calorimeter-collision-cascade where it comes to rest

$$W_{\text{cal}}: \quad \textcircled{a} * \textcircled{n}_{\mathbf{v'}} \quad \Rightarrow \quad \textcircled{a} * \textcircled{n}_{\mathbf{0}} \, , \; k \cdot \mathbf{1}_E\big|_{\mathbf{0}} \, , \; l \cdot \mathbf{1_p} \tag{63}$$

in return for extracting the reservoir balance, a corresponding number k equivalent energetic units $\mathbf{1}_E\big|_{\mathbf{0}}$ and l impulse units $\mathbf{1_p} := \textcircled{1}_{\mathbf{v_1}}$ into the direction of $\mathbf{v'}$. In our physical model for absorption action W_{cal} the (controlled) sequence of couplings and the (congruent) unit actions $w_{\mathbf{1}}$ are reversible (see Remark 9). Hence both calorimeter-mediated processes (61) and (63) are reversible.

We assume that binding action in system $\textcircled{a} * \textcircled{n}$ is *intrinsic* between objects $\textcircled{a}$ and $\textcircled{n}$. In the state of rest the bound system $\textcircled{a} * \textcircled{n}_{\mathbf{0}}$ can be reoriented into arbitrary direction θ

$$\mathrm{R}_\theta: \quad \textcircled{a} * \textcircled{n}_{\mathbf{0}} \quad \Rightarrow \quad \mathrm{R}_\theta \left[\textcircled{a} * \textcircled{n}_{\mathbf{0}}\right] \tag{64}$$

and by expending bounding energy E^* the direct inelastic collision $w_{(d)}$ can be reversed

$$\mathrm{R}_\theta \left[w_{(d)}^{-1}\right]: \quad \textcircled{a} * \textcircled{n}_{\mathrm{R}_\theta \mathbf{v'}} \quad \Rightarrow \quad \textcircled{a}_{\mathrm{R}_\theta \mathbf{v}_a} \, , \; \textcircled{n}_{\mathrm{R}_\theta \mathbf{v_1}} \tag{65}$$

so that unbound particles get kicked into corresponding direction θ. We essentially use the

- *isotropy* of unit action $w_{\mathbf{1}}^{(\theta)}$ as in (22) and
- construct a physical model from *separable* congruent unit actions $w_{\mathbf{1}}$ {3.1.1}.

We examine the diachronic association of these reversible interactions of motions

$$ {w_{(m)}}^{-1} \, * \, w_{(d)} \, * \, W_{\text{cal}} \, * \, \mathrm{R}_\theta \, * \, {W_{\text{cal}}^{(\theta)}}^{-1} \, * \, {w_{(d)}^{(\theta)}}^{-1} \, * \, w_{(m)}^{(\theta)}$$

in same objects $\textcircled{a}_{\mathbf{v}_a}$, $\textcircled{n}_{\mathbf{v_1}}$ resp. in same objects $\textcircled{a} * \textcircled{n}_{\mathbf{v'}}$ etc. which in between each coupling into consecutive actions move freely

$$\begin{aligned}
&\textcircled{a}_{\mathbf{0}} \, , \, \textcircled{n}_{\mathbf{0}} \, , \, \left(\tfrac{1}{2}\cdot n^2 + \tfrac{1}{2}\cdot n\right) \cdot \mathbf{1}_E\big|_{\mathbf{0}} \overset{(61)}{\Longrightarrow} \textcircled{a}_{\mathbf{v}_a} \, , \, \textcircled{n}_{\mathbf{v_1}} \overset{(62)}{\Longrightarrow} \textcircled{a} * \textcircled{n}_{\mathbf{v'}} \\
&\quad \overset{(63)}{\Longrightarrow} \textcircled{a} * \textcircled{n}_{\mathbf{0}} \, , \, k \cdot \mathbf{1}_E\big|_{\mathbf{0}} \, , \, l \cdot \textcircled{1}_{\mathbf{v_1}} \\
&\qquad \overset{(64)}{\Longrightarrow} \mathrm{R}_\theta \left[\textcircled{a} * \textcircled{n}_{\mathbf{0}}\right] \, , \, k \cdot \mathbf{1}_E\big|_{\mathbf{0}} \, , \, l \cdot \textcircled{1}_{\mathbf{v_1}} \, , \, \left(+\, l \cdot \textcircled{1}_{\mathrm{R}_\theta \mathbf{v_1}} - l \cdot \textcircled{1}_{\mathrm{R}_\theta \mathbf{v_1}}\right) \\
&\qquad\quad \overset{(63)}{\Longrightarrow} \textcircled{a} * \textcircled{n}_{\mathrm{R}_\theta \mathbf{v'}} \, , \, l \cdot \textcircled{1}_{\mathbf{v_1}} - l \cdot \textcircled{1}_{\mathrm{R}_\theta \mathbf{v_1}} \\
&\qquad\qquad \overset{(65)}{\Longrightarrow} \textcircled{a}_{\mathrm{R}_\theta \mathbf{v}_a} \, , \, \textcircled{n}_{\mathrm{R}_\theta \mathbf{v_1}} \, , \, l \cdot \textcircled{1}_{\mathbf{v_1}} - l \cdot \textcircled{1}_{\mathrm{R}_\theta \mathbf{v_1}} \\
&\qquad\qquad\quad \overset{(61)}{\Longrightarrow} \left(\tfrac{1}{2}\cdot n^2 + \tfrac{1}{2}\cdot n\right) \cdot \mathbf{1}_E\big|_{\mathbf{0}} \, , \, \textcircled{a}_{\mathbf{0}} \, , \, \textcircled{n}_{\mathbf{0}} \, , \quad l \cdot \textcircled{1}_{\mathbf{v_1}} - l \cdot \textcircled{1}_{\mathrm{R}_\theta \mathbf{v_1}} \quad .
\end{aligned}$$

Both objects $ⓐ_{\mathbf{0}}$, $ⓝ_{\mathbf{0}}$ act as catalyzers. Throughout the process we - temporarily - expend energetic units $\mathbf{1}_E\big|_{\mathbf{0}}$ and bounding energy E^* but in the end they are all recycled back into the reservoir. The association of (reversible) actions does not effect spectators $ⓐ_{\mathbf{0}}$, $ⓝ_{\mathbf{0}}$. In the end of this *circular process* we expend $-l$ impulse units $①_{\mathrm{R}_\theta \mathbf{v_1}}$ into arbitrarily rotated direction θ in return for generating l impulse units $①_{\mathbf{v_1}}$ without effecting anything else.

This hypothetical action would violate physical principles

- Principle of Sufficient Reason (Euler introduced this basic principle for Dynamics [2]) and
- Principle of the Impossibility of a Perpetuum Mobile.

Every change in the state motion requires an external cause (physical reason). A suitably moving observer (with velocity $\mathbf{v}'$) could set initially resting reservoir particles $\{①_{\mathbf{0}}\}$ into motion with unit velocity $\mathbf{v_1}$ but also into opposite direction with velocity $-\mathbf{v_1}$ without any effort. From each pair of antiparallel impulse units $\{①_{-\mathbf{v_1}} , ①_{\mathbf{v_1}}\}$ he could generate energetic units $\mathbf{1}_E\big|_{\mathbf{0}}$ which would provide (unlimited) source of energy without any reaction.

The violation of these physical principles stems from the hypothetical assumption $\mathbf{v}' \neq 0$. Therefore in mediated inelastic collision $w_{(m)}$ and in direct inelastic collision $w_{(d)}$ incident particle $ⓐ_{\mathbf{v}_a}$ and impulse model $ⓝ_{\mathbf{v_1}}$ must come to rest (independently of their unbound resp. bound final state). The form of absorbing bounding energy E^* has no effect on the impulse behavior. Our physical model for momentum $① * \ldots * ①_{\mathbf{v_1}} \;\sim_{\mathbf{p}}\; ⓐ_{\mathbf{v}_a}$ has same impulse behavior as incident particle $ⓐ_{\mathbf{v}_a}$.

□

In physical model W_{cal} for absorption of individual object $ⓐ_{\mathbf{v}_a}$ in a calorimeter reservoir the transferred momentum is conserved. A generic interaction of motion in a system with multiple elements $ⓘ_{\mathbf{v}_i}$ with $i = 1, \ldots, N$ can be modeled by separate absorption actions W_{cal} for individual elements (see Lemma 2). Therefore we conclude:

Corollary 3 *Momentum has a conserved quantity in a generic interaction of motion.*

Remark 12 *We justify the abstract quantity of energy and momentum (independently from the source). The calorimeter-collision-cascade W_{cal} (42) is a physical model for the absorption action w of individual particle $ⓐ_{\mathbf{v}_a}$ in a calorimeter*

$$W_{\mathrm{cal}} \;\sim_{E,\mathbf{p}}\; w \;.$$

The (controlled) replacement process W_{cal} redistributes - in an exactly elastic, reversible and practically instantaneous way - energy and impulse from incident object $ⓐ_{\mathbf{v}_a}$ onto equivalent dynamical units $\mathbf{1}_E\big|_{\mathbf{0}}$ and $①_{\mathbf{v_1}}$ from the reservoir.

In physical model W_{cal} (42) for absorption of one separate object $ⓐ_{\mathbf{v}_a}$ with velocity $\mathbf{v}_a$ we extract impulse units $①_{\mathbf{v_1}}$ and $①_{-\mathbf{v_1}}$ - on the left and right side of the calorimeter-collision-cascade - which are exactly antiparallel (see figure 13). In the calorimeter measurement W_{cal} of a generic interaction of motion the impulse units $①_{\mathrm{R}_{\alpha_1}\mathbf{v_1}}, \ldots, ①_{\mathrm{R}_{\alpha_N}\mathbf{v_1}}$ are extracted into arbitrary directions α_i for $i = 1, \ldots, N$.

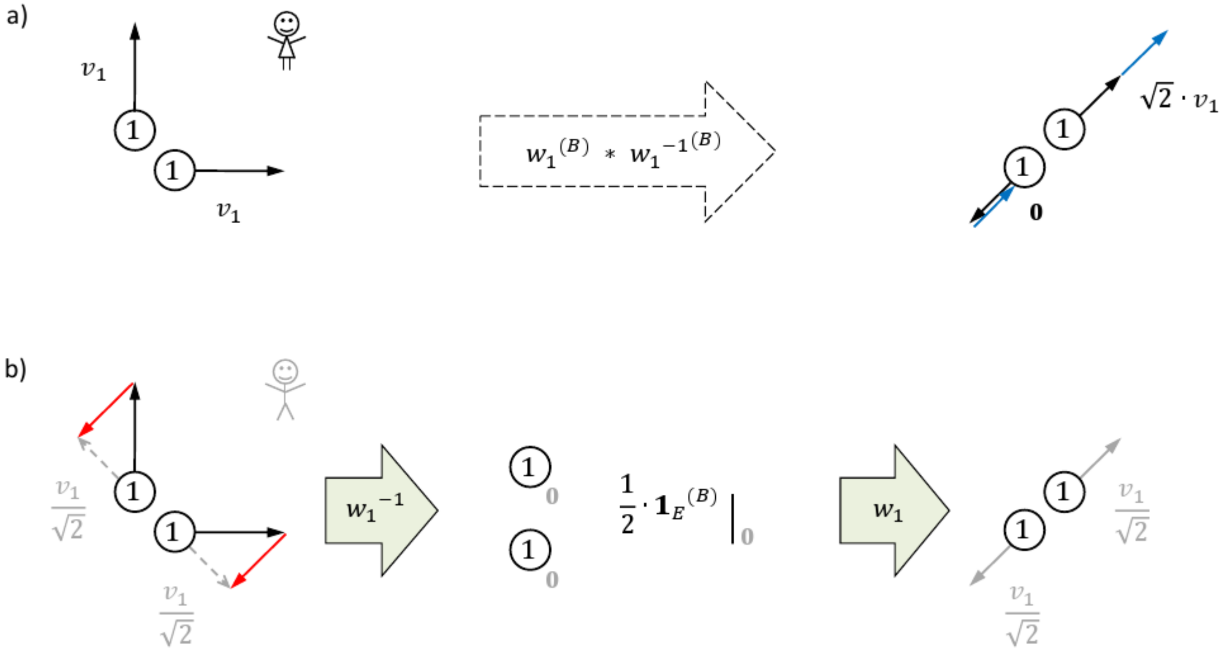

Figure 15: a) addition of impulse units for $\mathcal{A}$lice by b) reversible unit actions from $\mathcal{B}$ob

Theorem 3 *In a generic calorimeter measurement*[21] *a multitude of inequivalent impulse units* $\{\textcircled{1}_{\mathbf{v}_i}\}_{i=1\ldots N}$ *is extracted into different directions* $\mathbf{v}_i \neq \mathbf{v}_j$*. Direction and magnitude of the total momentum is calculable by* <u>*vectorial addition*</u>

$$\mathbf{p}\left[\{\textcircled{1}_{\mathbf{v}_1},\ldots,\textcircled{1}_{\mathbf{v}_N}\}\right] \;=\; \mathbf{p}\left[\textcircled{1}_{\mathbf{v}_1}\right]+\ldots+\;\mathbf{p}\left[\textcircled{1}_{\mathbf{v}_N}\right] \quad . \tag{66}$$

Proof: (In Galilei-Kinematics) we construct a physical model W for the absorption of a system with multiple elements $\textcircled{1}_{\mathbf{v}_i}$ with velocities $\mathbf{v}_i$ into various directions $\mathbf{v}_i \nparallel \mathbf{v}_j$ for elements $i \neq j \in 1,\ldots,N$

$$W: \quad \{\textcircled{1}_{\mathbf{v}_1},\ldots,\textcircled{1}_{\mathbf{v}_N}\}\;,\;\textcircled{1}_{\mathbf{0}} \;\;\Rightarrow\;\; \{\textcircled{1}_{\mathbf{0}},\ldots,\textcircled{1}_{\mathbf{0}}\}\;,\;\textcircled{1}_{\mathbf{v}_{(N)}} \quad .$$

All elements $i=1,\ldots,N$ of the system $\{\textcircled{1}_{\mathbf{0}},\ldots,\textcircled{1}_{\mathbf{0}}\}$ come to rest in the calorimeter. In return one initially resting unit object $\textcircled{1}_{\mathbf{v}_{(N)}}$ is kicked out of the calorimeter reservoir with velocity $\mathbf{v}_{(N)}$.

The basic physical model for complete momentum transfer from moving particle $\textcircled{a}_{\mathbf{v}_a}$ onto another moving particle $\textcircled{b}_{\mathbf{v}_b}$ is illustrated in figure 15. For $\mathcal{A}$lice two equivalent objects $\textcircled{1}_{\mathbf{v}_i}$ $i=1,2$ move with unit velocity into perpendicular directions

$$\mathbf{v}_1 = \begin{pmatrix}1\\0\end{pmatrix}\cdot\mathbf{v}_{\mathbf{1}(\mathcal{A})} \qquad \text{resp.} \qquad \mathbf{v}_2 = \begin{pmatrix}0\\1\end{pmatrix}\cdot\mathbf{v}_{\mathbf{1}(\mathcal{A})} \quad .$$

[21] Independently from the inertial mass m_a of absorbed particle $\textcircled{a}_{\mathbf{v}_a}$ the calorimeter-collision-cascade (42) generates a corresponding number of impulse units $\textcircled{1}_{\mathbf{v_1}}$ - from initially resting elements of the calorimeter reservoir $\{\textcircled{1}_{\mathbf{0}}\}$ - into the respective direction of motion of incident particle $\textcircled{a}_{\mathbf{v}_a}$.

Let $\mathcal{A}$lice move relative to $\mathcal{B}$ob with constant velocity $\mathbf{v}_{\mathcal{A}} = -\frac{1}{2} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}$. For $\mathcal{B}$ob both particles $i = 1, 2$ move into opposite direction with measured values

$$\mathbf{v}_i^{(\mathcal{B})} = \mathbf{v}_i^{(\mathcal{A})} + \mathbf{v}_{\mathcal{A}}^{(\mathcal{B})} = \pm \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} \qquad \text{with} \quad \mathbf{v}_{\mathcal{A}}^{(\mathcal{B})} = -\frac{1}{2} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and velocity $v_i^{(\mathcal{B})} = \frac{1}{\sqrt{2}} \cdot v_{\mathbf{1}^{(\mathcal{B})}}$.

Let $\mathcal{B}$ob absorb both particles in a calorimeter-collision-cascade W_{cal} where they come to rest

$$w_1: \quad ①_{-\frac{1}{\sqrt{2}} \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}}, ①_{\frac{1}{\sqrt{2}} \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}} \Rightarrow ①_{\mathbf{0}}, ①_{\mathbf{0}}, \frac{1}{2} \cdot \mathbf{1}_E^{(\mathcal{B})}\Big|_{\mathbf{0}}$$

in return for extracting the equivalent of $\frac{1}{2}$ energetic units $\mathbf{1}_E^{(\mathcal{B})}\big|_{\mathbf{0}}$ which he immediately expends in a consecutive calorimeter mediated action w_1^{-1} suitably rotated by 90° against both resting particles

$$\mathrm{R}_{90^\circ}\left[w_1^{-1}\right]: \quad ①_{\mathbf{0}}, ①_{\mathbf{0}}, \frac{1}{2} \cdot \mathbf{1}_E^{(\mathcal{B})}\Big|_{\mathbf{0}} \Rightarrow ①_{-\frac{1}{\sqrt{2}} \cdot \mathrm{R}_{90^\circ} \mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}}, ①_{\frac{1}{\sqrt{2}} \cdot \mathrm{R}_{90^\circ} \mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}}$$

where particles $i = 1, 2$ are kicked with opposite velocity $\mathbf{v}'_i = \pm\frac{1}{2} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}$ into the direction of $\mathcal{A}$lice. For her they have measured values of velocity $\mathbf{v'}_i^{(\mathcal{A})} = \mathbf{v'}_i^{(\mathcal{B})} + \mathbf{v}_{\mathcal{B}}^{(\mathcal{A})}$ with $\mathbf{v}_{\mathcal{B}}^{(\mathcal{A})} = \frac{1}{2} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\mathbf{v}'_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{A})}} \qquad \text{resp.} \qquad \mathbf{v}'_2 = 0 \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{A})}} \quad .$$

In $\mathcal{B}$ob's (elastic) association of reversible unit actions $w_1^{(\mathcal{B})} * w_1^{-1(\mathcal{B})}$ particle $①_{\mathbf{v}'_2}$ comes to rest $\mathbf{v'}_2^{(\mathcal{A})} = 0$. It has transferred all momentum (elastically) onto particle $①_{\mathbf{v}'_1}$ which moves with final velocity $\mathbf{v'}_1^{(\mathcal{A})} = \mathbf{v}_1^{(\mathcal{A})} + \mathbf{v}_2^{(\mathcal{A})}$.

For a system of N inequivalent impulse units $\{①_{\mathbf{v}_1}, ①_{\mathbf{v}_2}, \ldots, ①_{\mathbf{v}_N}\}$ $\mathcal{A}$lice successively transfers the impulse of all remaining particles $①_{\mathbf{v}_i}$ with $i = 2, \ldots, N$ onto particle $①_{\mathbf{v}_1}$. In the beginning of the induction let particle $①_{\mathbf{v}_1}$ have velocity $\mathbf{v}_1 =: \mathbf{v}_{(1)}$.

At each step $n \to n+1$ of the induction $\mathcal{A}$lice provides particle $①_{\mathbf{v}_{(n)}}$ with current velocity $\mathbf{v}_{(n)}$ and the next impulse unit $①_{\mathbf{v}_{n+1}}$ from the system. By a controlled sequence of (congruent) unit actions

$$W^{(n)}: \quad ①_{\mathbf{v}_{(n)}}, ①_{\mathbf{v}_{n+1}} \Rightarrow ①_{\mathbf{v}'_{(n)}}, ①_{\mathbf{0}}$$

we transfer all momentum from particle $①_{\mathbf{v}_{n+1}}$ onto particle $①_{\mathbf{v}'_{(n)}}$ and couple the latter - with velocity $\mathbf{v}_{(n+1)} := \mathbf{v}'_{(n)}$ - into the next step of the induction (see figure 16a).

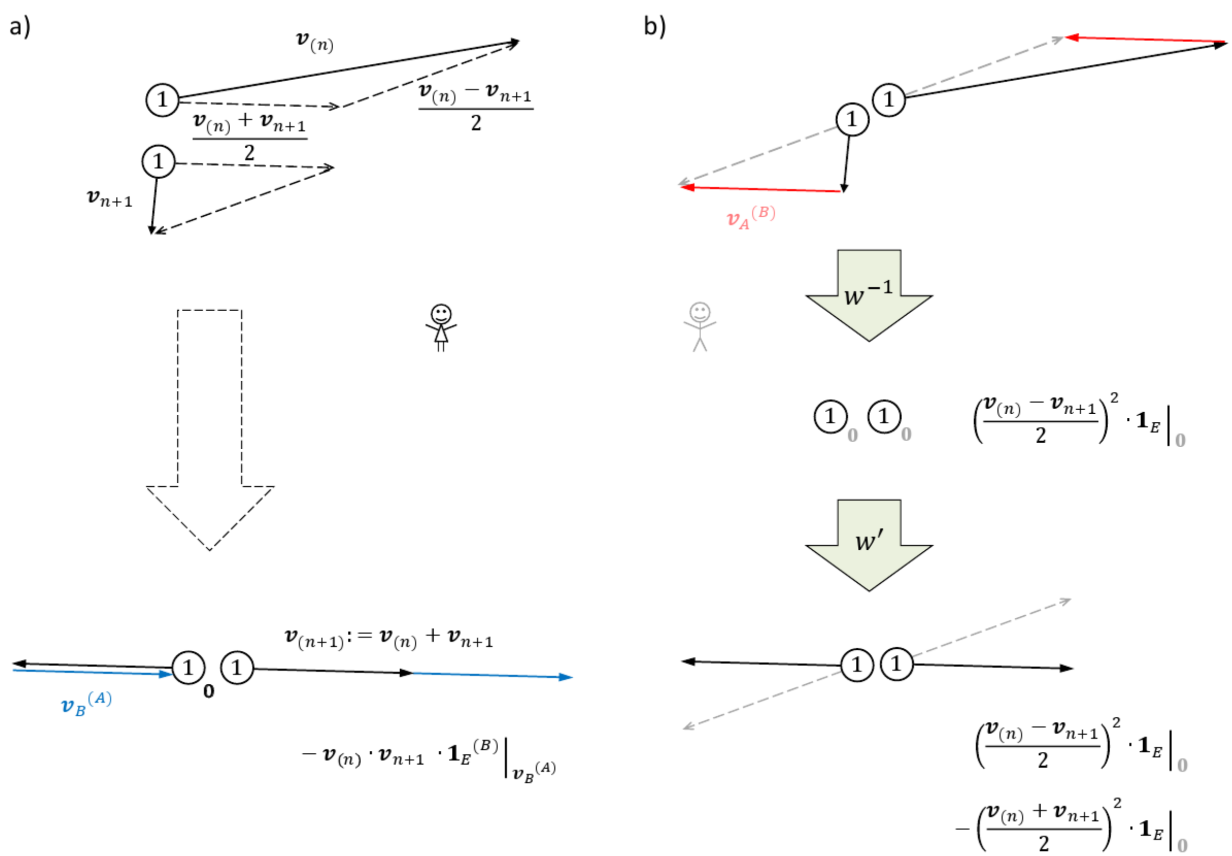

Figure 16: a) successive transfer of momentum from $①_{\mathbf{v}_{n+1}}$ to $①_{\mathbf{v}_{(n)}}$ for $\mathcal{A}$lice by means of b) congruent dynamical units of $\mathcal{B}$ob

Let $\mathcal{A}$lice conduct step $n \to n+1$ of the impulse-transfer process between the objects $①_{\mathbf{v}_{(n)}}$ and $①_{\mathbf{v}_{n+1}}$. Let $\mathcal{A}$lice move relative to $\mathcal{B}$ob with velocity $\mathbf{v}_{\mathcal{A}} = \mathbf{v}_{\mathcal{A}}^{(\mathcal{B})} \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}$ with measured value $\mathbf{v}_{\mathcal{A}}^{(\mathcal{B})} = -\frac{1}{2} \cdot \left(\mathbf{v}_{(n)}^{(\mathcal{A})} + \mathbf{v}_{n+1}^{(\mathcal{A})} \right)$.

For $\mathcal{B}$ob measured values of velocity transform Galilei-covariant (see Remark 8). Both particles $①_{\mathbf{v}_i}$ move antiparallel with measured values

$$\mathbf{v}_i^{(\mathcal{B})} = \mathbf{v}_i^{(\mathcal{A})} + \mathbf{v}_{\mathcal{A}}^{(\mathcal{B})} = \begin{cases} \frac{1}{2} \cdot \left(\mathbf{v}_{(n)}^{(\mathcal{A})} - \mathbf{v}_{n+1}^{(\mathcal{A})} \right) & , \quad i = (n) \\ -\frac{1}{2} \cdot \left(\mathbf{v}_{(n)}^{(\mathcal{A})} - \mathbf{v}_{n+1}^{(\mathcal{A})} \right) & , \quad i = n+1 \end{cases} .$$

$\mathcal{B}$ob absorbs both particles by calorimeter-mediated inelastic collision

$$w^{-1} : \quad ①_{\mathbf{v}_{(n)}} \,,\, ①_{\mathbf{v}_{n+1}} \overset{(48)}{\Rightarrow} ①_{\mathbf{0}} \,,\, ①_{\mathbf{0}} \,,\, \left(\frac{\mathbf{v}_{(n)}^{(\mathcal{A})} - \mathbf{v}_{n+1}^{(\mathcal{A})}}{2} \right)^2 \cdot \mathbf{1}_E^{(\mathcal{B})}\Big|_{\mathbf{0}}$$

in return for $\left(\frac{\mathbf{v}_{(n)}^{(\mathcal{A})} - \mathbf{v}_{n+1}^{(\mathcal{A})}}{2} \right)^2$ of $\mathcal{B}$ob's energetic units $\mathbf{1}_E^{(\mathcal{B})}\big|_{\mathbf{0}}$ (analogous to proof of Lemma 3). By expending $\left(\frac{\mathbf{v}_{(n)}^{(\mathcal{A})} + \mathbf{v}_{n+1}^{(\mathcal{A})}}{2} \right)^2$ energetic units $\mathbf{1}_E^{(\mathcal{B})}\big|_{\mathbf{0}}$ from his reservoir in inelastic action

$$w' : \quad ①_{\mathbf{0}} \,,\, ①_{\mathbf{0}} \,,\, \left(\frac{\mathbf{v}_{(n)} + \mathbf{v}_{n+1}}{2} \right)^2 \cdot \mathbf{1}_E^{(\mathcal{B})}\Big|_{\mathbf{0}} \Rightarrow ①_{-\frac{\mathbf{v}_{(n)} + \mathbf{v}_{n+1}}{2} \mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}} \,,\, ①_{\frac{\mathbf{v}_{(n)} + \mathbf{v}_{n+1}}{2} \mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}}$$

$\mathcal{B}$ob kicks both resting particles with antiparallel velocity $\mathbf{v}'_i = \pm \frac{\mathbf{v}_{(n)}^{(\mathcal{A})} + \mathbf{v}_{n+1}^{(\mathcal{A})}}{2} \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}$ into the direction of $\mathcal{A}$lice (see figure 16b). For her in the end they have measured values of velocity

$$\mathbf{v}'^{(\mathcal{A})}_i = \mathbf{v}'^{(\mathcal{B})}_i + \mathbf{v}_{\mathcal{B}}^{(\mathcal{A})} = \begin{cases} \mathbf{v}_{(n)}^{(\mathcal{A})} + \mathbf{v}_{n+1}^{(\mathcal{A})} \quad , \quad i = (n) \\ 0 \quad , \quad i = n+1 \end{cases} .$$

with $\mathbf{v}_{\mathcal{B}}^{(\mathcal{A})} = -\mathbf{v}_{\mathcal{A}}^{(\mathcal{B})} = \frac{1}{2} \cdot \left(\mathbf{v}_{(n)}^{(\mathcal{A})} + \mathbf{v}_{n+1}^{(\mathcal{A})} \right)$ (see figure 16a).

In $\mathcal{B}$ob's series of calorimeter-mediated inelastic collisions $w^{-1(\mathcal{B})} * w'^{(\mathcal{B})} =: W^{(n)}$ particle $①_{\mathbf{v}_{n+1}}$ transfers all its momentum to particle $①_{\mathbf{v}_{(n)}}$

$$W^{(n)} : \quad ①_{\mathbf{v}_{(n)}} \,,\, ①_{\mathbf{v}_{n+1}} \Rightarrow ①_{\mathbf{0}} \,,\, ①_{\mathbf{v}'_{(n)}} \,,\, -\mathbf{v}_{(n)}^{(\mathcal{A})} \cdot \mathbf{v}_{n+1}^{(\mathcal{A})} \cdot \mathbf{1}_E^{(\mathcal{B})}\Big|_{\mathbf{v}_{\mathcal{B}}}$$

at the expense of $\left(\frac{\mathbf{v}_{(n)}^{(\mathcal{A})} - \mathbf{v}_{n+1}^{(\mathcal{A})}}{2} \right)^2 - \left(\frac{\mathbf{v}_{(n)}^{(\mathcal{A})} + \mathbf{v}_{n+1}^{(\mathcal{A})}}{2} \right)^2 = -\mathbf{v}_{(n)}^{(\mathcal{A})} \cdot \mathbf{v}_{n+1}^{(\mathcal{A})}$ equivalent energetic units $\mathbf{1}_E^{(\mathcal{B})}\big|_{\mathbf{v}_{\mathcal{B}}} \sim_E \mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}}$ (see Lemma 4). At every step $n \to n+1$ of the induction particle $①_{\mathbf{v}_{n+1}}$ comes to rest while collision partner $①_{\mathbf{v}'_{(n)}}$ moves on with velocity $\mathbf{v}'_{(n)} = \mathbf{v}_{(n)} + \mathbf{v}_{n+1} =: \mathbf{v}_{(n+1)}$ into the next round of (organized) momentum transfer.

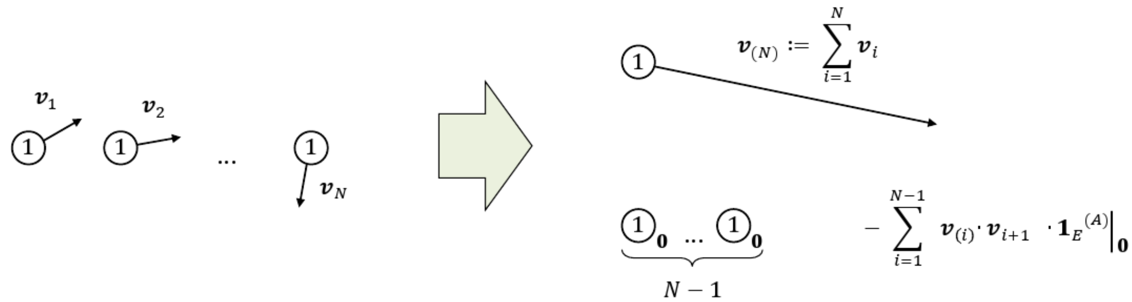

Figure 17: end effect - initial momentum of $\{①_{\mathbf{v}_2}, \ldots, ①_{\mathbf{v}_N}\}$ transferred to $①_{\mathbf{v}_1}$

After completing all induction steps of physical model $W^{(1)} * \ldots * W^{(N-1)}$:

$$①_{\mathbf{v}_1}\,,\ \{①_{\mathbf{v}_2}, \ldots, ①_{\mathbf{v}_N}\} \;\Rightarrow\; ①_{\mathbf{v}_{(N)}}\,,\ \{①_{\mathbf{0}}, \ldots, ①_{\mathbf{0}}\}\,,\ -\sum_{i=1}^{N-1} \mathbf{v}_{(i)} \cdot \mathbf{v}_{i+1} \cdot \mathbf{1}_E\big|_{\mathbf{0}} \tag{67}$$

all particles $i = 2, \ldots, N$ of the system come to rest while particle $①_{\mathbf{v}_{(N)}}$ moves on with final velocity $\mathbf{v}_{(N)} = \sum_{i=1}^{N} \mathbf{v}_i$ (see figure 17).

Energy and momentum of system $\{①_{\mathbf{v}_1}, \ldots, ①_{\mathbf{v}_N}\}$ is redistributed - in a reversible way - onto one particle $①_{\mathbf{v}_{(N)}}$ and

$$\begin{aligned}
-\sum_{i=1}^{N-1} \mathbf{v}_{(i)} \cdot \mathbf{v}_{i+1} &\overset{(67)}{=} -\sum_{i=1}^{N-1} \left(\sum_{k=1}^{i} \mathbf{v}_k \right) \cdot \mathbf{v}_{i+1} \;=\; -\sum_{k<i=1}^{N} \mathbf{v}_k \cdot \mathbf{v}_i \\
&= -\frac{1}{2} \cdot \sum_{i,k=1}^{N} \mathbf{v}_i \cdot \mathbf{v}_k \;+\; \frac{1}{2} \sum_{i=1}^{N} \mathbf{v}_i^2 \\
&= -\frac{1}{2} \cdot (\mathbf{v}_1 + \ldots + \mathbf{v}_N) \cdot (\mathbf{v}_1 + \ldots + \mathbf{v}_N) \;+\; \frac{1}{2} \sum_{i=1}^{N} \mathbf{v}_i^2
\end{aligned} \tag{68}$$

equivalent energetic units $\mathbf{1}_E\big|_{\mathbf{0}}$. In a calorimeter measurement W_{cal} the reservoir balance for interaction of motion (67)

$$\begin{aligned}
\text{RB}\left[W^{(1)} * \ldots * W^{(N-1)}\right] &= \text{RB}\left[①_{\mathbf{v}_1}, \ldots, ①_{\mathbf{v}_N} \;\Rightarrow\; ①_{\mathbf{0}}, \ldots, ①_{\mathbf{0}}\right] \\
&\quad -\text{RB}\left[①_{\mathbf{v}_{(N)}}\,,\ -\sum_{i=1}^{N-1} \mathbf{v}_{(i)} \cdot \mathbf{v}_{i+1} \cdot \mathbf{1}_E\big|_{\mathbf{0}} \;\Rightarrow\; ①_{\mathbf{0}}\right] \overset{!}{=} 0
\end{aligned} \tag{69}$$

decomposes into a sum of separate reservoir balances for absorbing respective objects in the initial and final state individually (see Lemma 2). The total reservoir balance has to vanish because a reversible action in a closed system cannot leave any mark in the (mediating) calorimeter model.

The total kinetic energy and momentum of system $\{\textcircled{1}_{\mathbf{v}_1}, \ldots, \textcircled{1}_{\mathbf{v}_N}\}$ becomes measurable

$$(E, \mathbf{p})\left[\textcircled{1}_{\mathbf{v}_1}, \ldots, \textcircled{1}_{\mathbf{v}_N}\right] \overset{(69)}{=} (E, \mathbf{p})\left[\textcircled{1}_{\mathbf{v}_{(N)}}, -\sum_{i=1}^{N-1} \mathbf{v}_{(i)} \cdot \mathbf{v}_{i+1} \cdot \mathbf{1}_E\Big|_{\mathbf{0}}\right]$$
$$\overset{(51)(58)(68)}{=} \left(\left\{\frac{1}{2}\sum_{i=1}^{N} \mathbf{v}_i^{(\mathcal{A})2}\right\} \cdot E_{\mathbf{1}^{(\mathcal{A})}}\ ,\ \left\{\mathbf{v}_1^{(\mathcal{A})} + \ldots + \mathbf{v}_N^{(\mathcal{A})}\right\} \cdot \mathbf{p}_{\mathbf{1}^{(\mathcal{A})}}\right)$$

by means of *equivalent* energetic units $\mathbf{1}_E\big|_{\mathbf{0}}$ and impulse units $\textcircled{1}_{\mathbf{v_1}}$ - which now all point into the <u>same direction</u> of $\mathbf{v}_{(N)} := \mathbf{v}_1 + \ldots + \mathbf{v}_N$. According to Lemma 6 momentum is conserved at each step of calorimeter mediated physical model (67). Hence the total momentum of system $\{\textcircled{1}_{\mathbf{v}_1}, \ldots, \textcircled{1}_{\mathbf{v}_N}\}$

$$\mathbf{p}^{(\mathcal{A})}\left[\{\textcircled{1}_{\mathbf{v}_1}, \ldots, \textcircled{1}_{\mathbf{v}_N}\}\right] \cdot \mathbf{p}_{\mathbf{1}^{(\mathcal{A})}} = \left\{\mathbf{p}^{(\mathcal{A})}\left[\textcircled{1}_{\mathbf{v}_1}\right] + \ldots + \mathbf{p}^{(\mathcal{A})}\left[\textcircled{1}_{\mathbf{v}_N}\right]\right\} \cdot \mathbf{p}_{\mathbf{1}^{(\mathcal{A})}}$$

is calculable by *vectorial sum of measured impulse values* $\mathbf{p}^{(\mathcal{A})}\left[\textcircled{1}_{\mathbf{v}_1}\right]$ of all elements $\textcircled{1}_{\mathbf{v}_i}$ $i = 1, \ldots, N$.

□

3.3 Inertial Mass

According to the Principle of Inertia moving bodies move on their own. Inertia is the resistance of an object $\textcircled{a}_{\mathbf{v}_a}$ against changes in its state of motion when under the influence of external causes. We can compare the inertial behavior of two objects $\textcircled{a}, \textcircled{b}$ directly in a physical ordering relation (see Definition 3). The <u>ordering relation</u> with regard to inertia is determined by the physical behavior in a collision: According to Galilei two objects have *same inertial mass*

$$\textcircled{a} \sim_{m^{(\text{inert})}} \textcircled{b}$$

if none of the two bodies overruns the other in an *inelastic* collision w where initially both objects move against one another with same velocity $\mathbf{v}_a \overset{!}{=} -\mathbf{v}_b$ [10]

$$w: \ \textcircled{a}_{\mathbf{v}}, \textcircled{b}_{-\mathbf{v}} \ \Rightarrow \ \textcircled{a} * \textcircled{b}_{\mathbf{0}} \ .$$

If the (bound) collision product $\textcircled{a} * \textcircled{b}_{\mathbf{v}'}$ continues moving into the direction of $\textcircled{a}_{\mathbf{v}}$ to the right, then the latter has more inertia $\textcircled{a} >_{m^{(\text{inert})}} \textcircled{b}$ than object $\textcircled{b}$ and vice versa. Let inelastic collision w be reversible. Then in an elastic (association of) collision $w_L := w * w^{-1}$ two bodies $\textcircled{a} \sim_{m^{(\text{inert})}} \textcircled{b}$ with same inertia

$$w_L: \ \textcircled{a}_{\mathbf{v}}, \textcircled{b}_{-\mathbf{v}} \ \Rightarrow \ \textcircled{a}_{-\mathbf{v}}, \textcircled{b}_{\mathbf{v}} \ . \tag{70}$$

must change their state of motion in the same antisymmetrical way.

We <u>associate</u> the inertial behavior of two objects $\textcircled{a}, \textcircled{b}$ by bounding '$*$' them together (e.g. by a practically massless sling). Then the bound composite $\textcircled{a} * \textcircled{b}$ acts like a single rigid body (see figure 1b). In a collision against generic object $\textcircled{n}_{\mathbf{v}_n}$

$$w: \quad \textcircled{a} * \textcircled{b}_{\mathbf{v}}\,,\ \textcircled{n}_{\mathbf{v}_n} \quad \Rightarrow \quad \textcircled{a} * \textcircled{b}_{\mathbf{v}'}\,,\ \textcircled{n}_{\mathbf{v}'_n} \ .$$

both elements $\textcircled{a}, \textcircled{b}$ of the composite act against *same* changes of their state of motion. Their *inertia* against external collision opponent $\textcircled{n}_{\mathbf{v}_n}$ is *combined*. We represent the <u>unit</u> (device) for inertia by the inertial behavior of $\mathcal{A}$lice standardized objects $\textcircled{1}$.

We construct a <u>physical model</u> for the inertia of object $\textcircled{a}$ by bounding together equivalent unit objects $\textcircled{1} * \ldots * \textcircled{1}$. The inertial mass of object $\textcircled{a}_{\mathbf{v}_a}$ becomes measurable by a tightly bound composite of equivalent unit objects

$$\textcircled{1} * \cdots * \textcircled{1} \ \sim_{m^{(\text{inert})}} \ \textcircled{a}$$

with same inertial behavior as incident object $\textcircled{a}_{\mathbf{v}_a}$. Both are interchangeable in collision action (70). In her physical model $\mathcal{A}$lice can count the number of equivalent unit elements $\textcircled{1}$. By the congruence principle the inertial mass of object $\textcircled{a}_{\mathbf{v}_a}$ is quantified by the number $m_a^{(\mathcal{A})}$ of unit elements $\textcircled{1}$

$$m\,[\textcircled{a}] =: m_a^{(\mathcal{A})} \cdot m_{\mathbf{1}^{(\mathcal{A})}}$$

and its unit mass $m_{\mathbf{1}^{(\mathcal{A})}} := m^{(\text{inert})}\,[\textcircled{1}]$. If particle $\textcircled{a}_{\mathbf{v}_a}$ is equivalent with unit object $\textcircled{1}$ this material constant $m_a^{(\mathcal{A})} \equiv 1$. Again this method of metrization is universally *reproducible* in an observer independent way.

Remark 13 *The quantity of inertial mass of particle* $\textcircled{a}_{\mathbf{v}_a}$ *corresponds to <u>amount of matter</u> in the equivalent physical model* $\textcircled{1} * \cdots * \textcircled{1} \ \sim_{m^{(\text{inert})}} \ \textcircled{a}$ *.*[22]

Our physical model W_{cal} for the absorption action of generic object $\textcircled{a}_{\mathbf{v}_a}$ in a calorimeter reservoir with equivalent unit objects $\{\textcircled{1}_{\mathbf{0}}\}$ makes kinetic energy and momentum measurable. The number of extracted (equivalent) energetic units and impulse units for absorbing generic object $\textcircled{a}_{\mathbf{v}_a}$ is the same as for absorbing an equally moving composite of unit objects $\textcircled{1} * \cdots * \textcircled{1} \sim_{m^{(\text{inert})}} \textcircled{a}_{\mathbf{v}_a}$ with same inertial mass (see Lemma 5). Kinetic energy and momentum of composite $\textcircled{1} * \cdots * \textcircled{1}$ are proportional to the amount of matter $\textcircled{1}$ it contains (see Lemma 2). Hence kinetic energy and momentum (of object $\textcircled{a}_{\mathbf{v}_a}$) are proportional to amount of matter (in the equivalent physical model) resp. to inertial mass. By means of the calorimeter-collision-cascade the interrelation between various aspects of an action: energy, momentum and inertial mass becomes transparent.

[22] The current mass standard is defined by one prototype $\mathbf{1}_{\text{kg}}$ - the Ur-Kilogram which is available in only one French laboratory. In recent approaches this mass unit is replaced by one Si atom $\mathbf{1}_{\text{Si}}$.

For *practical feasibility* one redefines the Kilogram by means of - a certain number N of - Si atoms: $1\text{kg} := N \cdot \mathbf{1}_{\text{Si}}$. Still each atom represents the inertial behavior of the mass unit in interactions of motion. In a *practical realization* of the Atom-counting approach one manufactures a single-crystal sphere of silicon atoms. Its radius (uncertainty on roughly a single atomic layer) and the lattice spacing between individual Si atoms (by X-ray spectroscopy) are precisely known. This *reproducible* prototype is among the roundest *man-made* objects in the world [34].

3.4 Energy

Leibniz introduced the principle to measure energy by its effect. We follow Schlaudt who has explained Leibniz method - how to quantify pre-theoretic notion of kinetic energy [23].

3.4.1 Kinetic Energy

Mach characterizes the everyday pre-scientific notion of 'driving force' (Latin: 'vis viva'): Soon after Galilei one did notice that behind the velocity of an object there is a certain capability to cause actions. Something which allows to overcome force. How to measure this 'something' was the subject of the 'vis viva' dispute [7]. That 'moving power' was initially a vague, pre-theoretic notion. It has the peculiar feature - Schlaudt explains - that it cannot be quantified directly but solely by means of the effect which it is capable to realize. This is not a mathematical problem - Schlaudt emphasizes - but a practical whose solution entails the mathematical expression for force.

Definition 8 *Kinetic energy* $E_{\text{kin}}\left[\textcircled{a}_{\mathbf{v}_a}\right]$ *is the potential to cause actions which is associated with decelerating a moving body* $\textcircled{a}_{\mathbf{v}_a}$*.*

Leibniz explains how to quantify the pre-theoretic notion of kinetic energy. In his conception the *measurement principles*

- equipollence
- congruence

come into play. Leibniz explains: *'il faut avoir recours à l'equipollence de la cause et de l'effect'* - i.e. one has to assess the cause by its effect. To measure the cause $\mathcal{U}$ (German: $\mathcal{U}$rsache) by its effect requires: (i) providing a precise standard action which successively consumes the cause $\mathcal{U}$, (ii) the cumulative effect of formal repetition and coupling of standard actions reproduces the action of $\mathcal{U}$ and (iii) guarantee that all copies of the standard action are equivalent with one another [23]. In a practical test those copies have to be equivalent with regard to pre-theoretic ordering relations $\sim_{E,\mathbf{p}}$ {2.3}. Leibniz presents various candidates for standard actions, including the compression of a standard spring by a fixed standard length (see d'Alembert's example {3.1.1}). In this physical model the cumulative effect is achieved by repeating the same action of equivalent springs - i.e. by associating congruent standard actions $w_{\mathbf{1}}$. The effect of the kinetic energy of moving body $\textcircled{a}_{\mathbf{v}_a}$ is measured *directly* by repeating the same unit action (of a standard spring). In this physical model one can count the number of springs which can be compressed while moving object $\textcircled{a}_{\mathbf{v}_a}$ comes to rest.

According to the principle of inertia moving bodies move on their own. Their state of motion is preserved unless they are effected by an external cause. The kinetic energy of *individual particle* $\textcircled{a}_{\mathbf{v}_a}$ is associated with changes in state of motion $\mathbf{v}_a$ and the effect against other elements of the interacting system $\textcircled{a} \cup \textcircled{1} \cup \ldots \cup \textcircled{n}$. We can compare the energetic behavior of two moving particles $\textcircled{a}_{\mathbf{v}_a}, \textcircled{b}_{\mathbf{v}_b}$ directly in a physical ordering relation (see Definition 4). The ordering relation with regard to kinetic energy is determined by the physical behavior in an absorption action: Moving particle $\textcircled{a}_{\mathbf{v}_a}$ has *same energetic* behavior as moving particle $\textcircled{b}_{\mathbf{v}_b}$

$$\textcircled{a}_{\mathbf{v}_a} \quad \sim_E \quad \textcircled{b}_{\mathbf{v}_b}$$

if the potential effect of absorbing particle $\textcircled{a}_{\mathbf{v}_a}$ in a system $\textcircled{1} \cup \ldots \cup \textcircled{n}$ is equivalent to the potential effect of absorbing particle $\textcircled{b}_{\mathbf{v}_b}$ - if coupled into the same system $\textcircled{1} \cup \ldots \cup \textcircled{n}$. Then by the equipollence of cause and effect the two moving particles $\textcircled{a}_{\mathbf{v}_a}, \textcircled{b}_{\mathbf{v}_b}$ are equivalent with regard to kinetic energy. Both particles can substitute one another in the comparison method with regard to their absorption effect. Under the abstraction 'energy' we regard individual sources solely as interchangeable representatives of their common quality: 'potential to cause action' (see Remark 6). If the potential effect of absorbing particle $\textcircled{a}_{\mathbf{v}_a}$ *exceeds* the potential effect of absorbing particle $\textcircled{b}_{\mathbf{v}_b}$ - e.g. when coupled into the same calorimeter reservoir $\{\textcircled{1}_{\mathbf{0}}\}$ - then particle $\textcircled{a}_{\mathbf{v}_a}$ has more kinetic energy $\textcircled{a}_{\mathbf{v}_a} >_E \textcircled{b}_{\mathbf{v}_b}$ than particle $\textcircled{b}_{\mathbf{v}_b}$ and vice versa.

We associate the kinetic energy of two moving particles $\textcircled{a}_{\mathbf{v}_a}$ and $\textcircled{b}_{\mathbf{v}_b}$ by coupling two consecutive absorption actions: We expend the kinetic energy of both objects against the same external (system of) object $\textcircled{I}_{\mathbf{v}}$ (with collective index $\mathrm{I} := 1, \ldots, N$ for all elements of the system)

$$\textcircled{a}_{\mathbf{v}_a}\,,\; \textcircled{b}_{\mathbf{v}_b}\,,\; \textcircled{I}_{\mathbf{v}_\mathrm{I}} \quad \Rightarrow \quad \textcircled{a}_{\mathbf{0}}\,,\; \textcircled{b}_{\mathbf{0}}\,,\; \textcircled{I}_{\mathbf{v}'_\mathrm{I}}$$

such that in the final state both objects $\textcircled{a}_{\mathbf{0}}$ and $\textcircled{b}_{\mathbf{0}}$ come to rest (as for the impulse behavior in {3.2}). We represent the energetic unit by the energetic behavior of $\mathcal{A}$lice standardized sources of energy $\mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}}$ (e.g. her standard springs) in her unit action $w_{\mathbf{1}^{(\mathcal{A})}}$.

We provide a physical model for the kinetic energy of particle $\textcircled{a}_{\mathbf{v}_a}$ from the calorimeter-collision-cascade W_{cal}. It reproduces the (reversible) absorption action of incident object $\textcircled{a}_{\mathbf{v}_a}$ against a calorimeter reservoir $\{\textcircled{1}_{\mathbf{0}}\}$. The physical model W_{cal} solely consists of congruent unit actions $w_{\mathbf{1}^{(\mathcal{A})}}$. In return for absorbing particle $\textcircled{a}_{\mathbf{v}_a}$ the reservoir extracts

$$\mathrm{RB}\left[\textcircled{a}_{\mathbf{v}_a} \Rightarrow \textcircled{a}_{\mathbf{0}}\right] \quad = \quad k_a \cdot \mathbf{1}_E\big|_{\mathbf{0}}\,,\; l_a \cdot \mathbf{1}_{\mathbf{p}}$$

k_a energetic units $\mathbf{1}_E\big|_{\mathbf{0}}$ and l_a momentum units $\mathbf{1}_{\mathbf{p}}$ (see Definition 7). By the equipollence of cause and effect the calorimeter extract

$$\mathrm{RB}\left[\textcircled{a}_{\mathbf{v}_a} \Rightarrow \textcircled{a}_{\mathbf{0}}\right] \quad \sim_E \quad \textcircled{a}_{\mathbf{v}_a}$$

has same energetic behavior (14) as incident particle $\textcircled{a}_{\mathbf{v}_a}$. Its kinetic energy is redistributed - in an exactly elastic and reversible way -

$$\begin{aligned} E_{\mathrm{kin}}\left[\textcircled{a}_{\mathbf{v}_a}\right] \quad &= \quad E\left[\mathrm{RB}\left[\textcircled{a}_{\mathbf{v}_a} \Rightarrow \textcircled{a}_{\mathbf{0}}\right]\right] \\ &\overset{(46)(51)}{=} \quad E\left[k_a \cdot \mathbf{1}_E\big|_{\mathbf{0}}\right] \;+\; E\left[l_a \cdot \textcircled{1}_{\mathbf{v_1}}\right] \end{aligned} \tag{71}$$

onto $k_a \overset{(41)(56)}{:=} \frac{1}{2} \cdot \frac{m_a}{m_{\mathbf{1}}} \cdot \left(\left(\frac{\mathbf{v}_a}{\mathbf{v_1}}\right)^2 - \frac{\mathbf{v}_a}{\mathbf{v_1}}\right)$ energetic units $\mathbf{1}_E\big|_{\mathbf{0}}$ and $l_a \overset{(41)(56)}{:=} \frac{m_a}{m_{\mathbf{1}}} \cdot \frac{\mathbf{v}_a}{\mathbf{v_1}}$ impulse units $\mathbf{1}_{\mathbf{p}} := \textcircled{1}_{\mathbf{v_1}}$ from the initially resting reservoir $\{\textcircled{1}_{\mathbf{0}}\}$ (see Remark 12). That output of the calorimeter reservoir is our physical model for kinetic energy. In her physical model $\mathcal{A}$lice can count the number of equivalent dynamical units $\mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}}$ resp. $\mathbf{1}_{\mathbf{p}}^{(\mathcal{A})}$. By the congruence principle the kinetic energy of individual particle $ⓐ_{\mathbf{v}_a}$ becomes measurable by the number $E_a^{(\mathcal{A})}$ of extractable energetic units $\mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}}$

$$E\left[ⓐ_{\mathbf{v}_a}\right] =: E_a^{(\mathcal{A})} \cdot E_{\mathbf{1}^{(\mathcal{A})}}$$

and its unit energy $E_{\mathbf{1}^{(\mathcal{A})}} := E\left[\mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}}\right]$. Again this method of quantification is universally *reproducible* in an observer independent way.

3.4.2 Potential Energy

We grasp an action impartially as the collective behavior of an interacting system. During an interaction of motion elements in system $G_1 \cup \ldots \cup G_N$ act against one another {2.2}. Each object G_i acts against changes in its state of motion. In isolation elements would preserve their state of motion.

Remark 14 *The system includes all elements $G_1, \ldots, G_N$ and* causes $\mathcal{U}$ *of interaction which are (i) implicit by their presence in a (bound) system (German: Systemdasein) or (ii) explicit by coupling an external source of energy $\mathcal{U}_E$.*

We determine the energy and momentum of the entire system by its physical behavior in an absorption resp. collision action against an external calorimeter.

Definition 9 *The potential energy of - configuration transitions in a - conservative system $\{G_I\}$ is associated with the kinetic effect, i.e. a change in the state of motion of all elements $G_1, \ldots, G_N$ {4.2}.*

As introductory example we examined the action in a system with two elements $ⓐ \cup ⓑ$. In direct inelastic collision (62) incident particles $ⓐ_{\mathbf{v}_a}$ and $ⓑ_{\mathbf{v}_b}$

$$w_{(d)}: \quad ⓐ_{\mathbf{v}_a},\ ⓑ_{\mathbf{v}_b} \quad \Rightarrow \quad ⓐ * ⓑ_{\mathbf{0}}$$

form a bound object $ⓐ * ⓑ_{\mathbf{0}}$ in the state of rest (where '$*$' symbolizes the inner bounding). In this simple example we distinguish two configurations of the system: (i) in the initial state both elements $ⓐ, ⓑ$ are isolated from one another and (ii) in the final state they exist as a bound system $ⓐ * ⓑ$. In the combined process with (calorimeter) mediated inelastic collision $w_{(m)}$ (61) two intially resting and isolated elements $ⓐ_{\mathbf{0}}, ⓑ_{\mathbf{0}}$

$$w_{(m)}^{-1} * w_{(d)}: \quad ⓐ_{\mathbf{0}},\ ⓑ_{\mathbf{0}},\ k \cdot \mathbf{1}_E\big|_{\mathbf{0}} \quad \Rightarrow \quad ⓐ * ⓑ_{\mathbf{0}}$$

form a bound state $ⓐ * ⓑ_{\mathbf{0}}$ in return for expending k energetic units $\mathbf{1}_E\big|_{\mathbf{0}}$. It takes bounding energy E^* to change the configuration from isolated elements $ⓐ, ⓑ \Rightarrow ⓐ * ⓑ$ into a bound state. If direct inelastic collision is *reversible* then corresponding bounding energy E^*

$$w_{(d)}^{-1}: \quad ⓐ * ⓑ_{\mathbf{0}} \quad \Rightarrow \quad ⓐ_{\mathbf{v}_a},\ ⓑ_{\mathbf{v}_b}$$

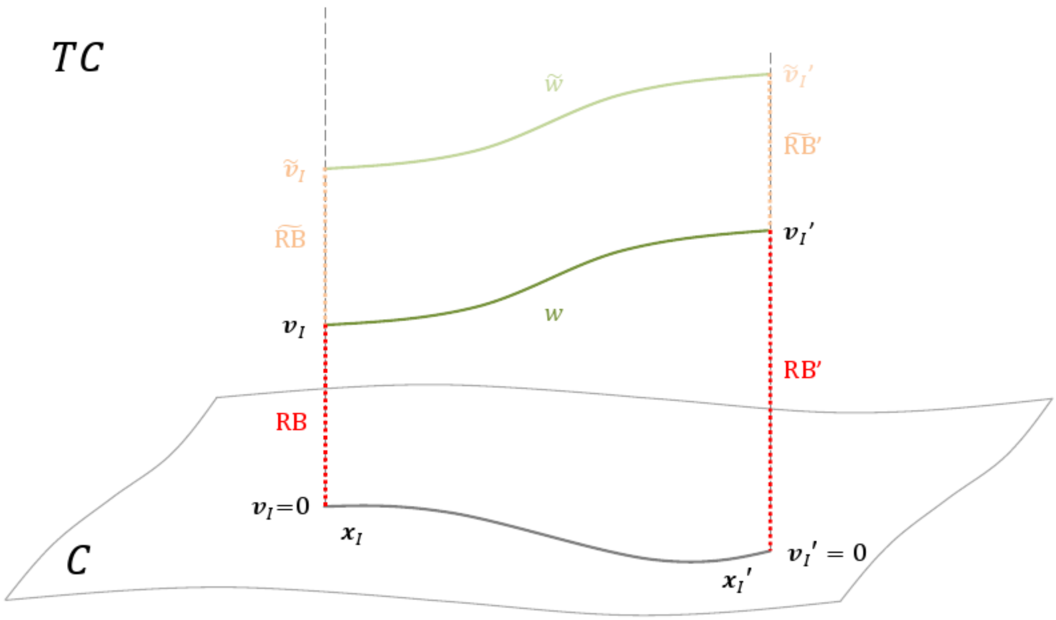

Figure 18: potential energy (and momentum) extract from configuration transitions

can be transferred into kinetic energy of unbound elements $\textcircled{a}_{\mathbf{v}_a}, \textcircled{b}_{\mathbf{v}_b}$. That potential energy $E^* := V\,[\textcircled{a} * \textcircled{n}_{\mathbf{0}} \Rightarrow \textcircled{a}_{\mathbf{0}}, \textcircled{b}_{\mathbf{0}}]$ of system $\textcircled{a} \cup \textcircled{b}$ is associated with the *transition* from a bound to an unbound configuration. For reversible actions we quantify the potential energy of - configurations in a - bound system $\textcircled{a} \cup \textcircled{b}$ by the potential kinetic effect from separating its elements $\textcircled{a}, \textcircled{b}$ in a calorimeter-measurement.

In conservative systems $\textcircled{1} \cup \ldots \cup \textcircled{n}$ the potential energy (83) is solely determined by the configuration $\mathbf{x}_I := \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of its elements $\textcircled{i}$ (collective index $I := \{1, \ldots, n\}$ denotes all elements of the system). A transition in the collective configuration $\mathbf{x}_I \Rightarrow \mathbf{x}'_I$

$$w: \quad \textcircled{1} \cup \ldots \cup \textcircled{n}_{\mathbf{x}_I, \mathbf{v}_I} \quad \Rightarrow \quad \textcircled{1} \cup \ldots \cup \textcircled{n}_{\mathbf{x}'_I, \mathbf{v}'_I}$$

is associated with changing state of motion $\mathbf{v}_I$ for respective elements $I = 1, \ldots, n$ of the system. We mediate the transition between two resting configurations $\mathbf{x}_I \Rightarrow \mathbf{x}'_I$ of the system

$$\mathrm{RB}^{-1}\Big|_{\mathbf{x}_I} * w * \mathrm{RB}'\Big|_{\mathbf{x}'_I}: \quad \textcircled{I}_{\mathbf{x}_I, \mathbf{v}_I = \mathbf{0}} \quad \Rightarrow \quad \textcircled{I}_{\mathbf{x}'_I, \mathbf{v}'_I = \mathbf{0}} \tag{72}$$

by coupling three physical processes: calorimeter intervention $\mathrm{RB}^{-1}\big|_{\mathbf{x}_I}$ at configuration $\mathbf{x}_I$ prepares the initial state of motion for consecutive action w. Its kinetic effect is extracted by a consecutive calorimeter measurement $\mathrm{RB}'\big|_{\mathbf{x}'_I}$ in the final configuration $\mathbf{x}'_I$ (see lower square in figure 18) so that after completion all elements of the system remain at rest.

Assume we can conduct calorimeter measurements $\mathrm{RB}\,[\textcircled{i}_{\mathbf{v}_i} \Rightarrow \textcircled{i}_{\mathbf{0}}]\,\big|_{\mathbf{x}_i}$ for each element $\textcircled{i}$ of the system at its initial $\mathbf{x}_i$ and final location $\mathbf{x}'_i$ in a practically instantaneous way {4.1}. We measure separate reservoir balances:

- for absorbing the collective motion $\mathbf{v}_I := \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \Rightarrow \mathbf{0}$ of all elements $①_{\mathbf{v}_1}, \ldots, ⓝ_{\mathbf{v}_n}$ at the initial configuration $\mathbf{x}_I := \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of the system and
- for absorbing the motion $\mathbf{v}'_I \Rightarrow \mathbf{0}$ of all elements at the final configuration $\mathbf{x}'_I$.

We illustrate the extraction (resp. expenditure) of kinetic energy and momentum from individual elements of the system by lowering (resp. rising) their actual state of motion $\mathbf{v}_I$ along vertical fibres $T_{\mathbf{x}_I}\mathcal{C}$ in velocity space (see figure 18).

We extract the kinetic effect for transitions between resting configurations

$$\begin{aligned}\mathrm{RB}\left[①_{\mathbf{x}_I,\mathbf{v}_I=\mathbf{0}} \Rightarrow ①_{\mathbf{x}'_I,\mathbf{v}'_I=\mathbf{0}}\right] &\overset{(72)}{=} \mathrm{RB}\left[①_{\mathbf{x}_I,\mathbf{0}} \Rightarrow ①_{\mathbf{x}_I,\mathbf{v}_I} \overset{w}{\Rightarrow} ①_{\mathbf{x}'_I\mathbf{v}'_I} \Rightarrow ①_{\mathbf{x}'_I,\mathbf{0}}\right] \\ &= \mathrm{RB}\left[①_{\mathbf{x}_I,\mathbf{0}} \Rightarrow ①_{\mathbf{x}_I,\mathbf{v}_I}\right] + \underbrace{\mathrm{RB}\left[①_{\mathbf{x}_I,\mathbf{v}_I} \Rightarrow ①_{\mathbf{x}'_I\mathbf{v}'_I}\right]}_{=0} + \mathrm{RB}\left[①_{\mathbf{x}'_I\mathbf{v}'_I} \Rightarrow ①_{\mathbf{x}'_I,\mathbf{0}}\right] \\ &= -\,\mathrm{RB}\left[①_{\mathbf{v}_I} \Rightarrow ①_{\mathbf{0}}\right]\Big|_{\mathbf{x}_I} + \mathrm{RB}\left[①_{\mathbf{v}'_I} \Rightarrow ①_{\mathbf{0}}\right]\Big|_{\mathbf{x}'_I} \,. \end{aligned} \tag{73}$$

where *free* action $\mathrm{RB}[w] \overset{!}{=} 0$ evolves isolated from exterior *steering actions* (e.g. calorimetric interventions by physicists {4.1}) without accounting for energetic and impulse units coupled into the system. Superposition Principle and compatibility assumptions {4.2} admit intrinsic absorption measurements on individual elements independent from their bound state $\mathbf{x}_I$, $\mathbf{x}'_I$ as if each were isolated from the system. Then potential gain from configuration transitions

$$\mathrm{RB}\left[①_{\mathbf{x}_I,\mathbf{0}} \Rightarrow ①_{\mathbf{x}'_I,\mathbf{0}}\right] \overset{(73)}{=} -\,\mathrm{RB}\left[①_{\mathbf{v}_I} \Rightarrow ①_{\mathbf{v}'_I}\right] \tag{74}$$

matches the expense of steering actions which reproduce the kinetic changes of action w on separated elements (see lower square in figure 18). In action w the potential cause $\mathbf{x}_I \Rightarrow \mathbf{x}'_I$ and the kinetic effect $\mathbf{v}_I \Rightarrow \mathbf{v}'_I$ are in opposition to one another.

For *reversible* interactions of motion w in a *closed system* the total number of extracted dynamical units $\mathbf{1}_E\big|_{\mathbf{0}}$ and $\mathbf{1}_{\mathbf{p}}$ is independent from individual action $w : \mathbf{x}_I, \mathbf{v}_I \Rightarrow \mathbf{x}'_I, \mathbf{v}'_I$ resp. $\tilde{w} : \mathbf{x}'_I, \tilde{\mathbf{v}}'_I \Rightarrow \mathbf{x}_I, \tilde{\mathbf{v}}_I$. As both actions transfer - under initial conditions $\mathbf{v}_I$ resp. $\tilde{\mathbf{v}}'_I$ - between same configurations $\mathbf{x}_I \Leftrightarrow \mathbf{x}'_I$ of the system they can be associated into a *circular process*

$$\left(\tilde{\mathrm{RB}} - \mathrm{RB}\right) * w * \left(-\tilde{\mathrm{RB}}' + \mathrm{RB}'\right) * \tilde{w}$$

by means of calorimeter measurements $\mathrm{RB}\big|_{\mathbf{x}_I}$ resp. $\mathrm{RB}'\big|_{\mathbf{x}'_I}$ (see upper square in figure 18). The total calorimeter output $\tilde{\mathrm{RB}} - \mathrm{RB} \overset{!}{=} -\left(\tilde{\mathrm{RB}}' - \mathrm{RB}'\right)$ must vanish by the principle of impossibility of a Perpetuum Mobile. The extractable kinetic effect from configuration transitions (73) is independent from initial conditions under which free action w evolves.

By the equipollence principle the potential energy from *configuration transitions* $\mathbf{x}_I \Rightarrow \mathbf{x}'_I$ becomes measurable by the kinetic effect $\mathbf{v}_I \Rightarrow \mathbf{v}'_I$ on individual elements. In a conservative system {4.2} we determine potential energy

$$\begin{aligned} V_{\mathrm{pot}}\left[①_{\mathbf{x}_I} \Rightarrow ①_{\mathbf{x}'_I}\right] &:= E\left[\mathrm{RB}\left[①_{\mathbf{x}_I,\mathbf{0}} \Rightarrow ①_{\mathbf{x}'_I,\mathbf{0}}\right]\right] \\ &\overset{(74)}{=} -\,E\left[\mathrm{RB}\left[①_{\mathbf{v}_I} \Rightarrow ①_{\mathbf{v}'_I}\right]\right] = -\,E_{\mathrm{kin}}\left[①_{\mathbf{v}_I} \Rightarrow ①_{\mathbf{v}'_I}\right] \end{aligned} \tag{75}$$

from the acquired kinetic energy $E_{\mathrm{kin}}\left[\textcircled{I}_{\mathbf{v}_I}\right] = E_{\mathrm{kin}}\left[\textcircled{1}_{\mathbf{v}_1}\right] + \ldots + E_{\mathrm{kin}}\left[\textcircled{n}_{\mathbf{v}_n}\right]$ of all elements - which is quantified by our physical model for calorimetric measurements W_{cal}. According to this measurement principle total energy of the closed system is *conserved*

$$V_{\mathrm{pot}}\left[\textcircled{I}_{\mathbf{x}_I}\right] \;+\; E_{\mathrm{kin}}\left[\textcircled{I}_{\mathbf{v}_I}\right] \;\overset{(75)(71)}{=}\; V_{\mathrm{pot}}\left[\textcircled{I}_{\mathbf{x}'_I}\right] \;+\; E_{\mathrm{kin}}\left[\textcircled{I}_{\mathbf{v}'_I}\right] \;.^{23}$$

3.4.3 Quantification scheme

According to the equipollence principle the cause of an action is energetically equivalent to its effect. For the quantification of energy we construct *physical models for generic interactions* of motion w which reproduce the kinetic effect

$$w_{\mathbf{1}} \qquad \hookrightarrow \qquad W_{\mathrm{cal}}^{(i)} \qquad \hookrightarrow \qquad {W_{\mathrm{cal}}^{(I)}}^{-1} * w * W_{\mathrm{cal}}^{(I)} \quad . \tag{76}$$

As measurement standard we provide congruent unit actions $w_{\mathbf{1}}$ (e.g. the compression of a standard spring by a standard length). Despite complete abstraction from the inner dynamics $w_{\mathbf{1}}$ provides a precise kinetic effect (standard objects $\textcircled{1}$ are kicked into unit velocity $\mathbf{v}_{\mathbf{1}}$ (17)). By associating those unit actions - in a controlled way - we construct a physical model $W_{\mathrm{cal}}^{(i)}$ for the absorption action of individual particle $\textcircled{i}_{\mathbf{v}_i}$ with velocity $\mathbf{v}_i$ in a calorimeter $\{\textcircled{1}_{\mathbf{0}}\}$. And finally repeated calorimeter measurements for each element $\textcircled{i}$ $i \in I = 1, \ldots, n$ at the initial $\mathbf{x}_I$ and final $\mathbf{x}'_I$ configuration of the system separate the kinetic effect of generic interaction of motion w.

Energy (and momentum) is inseparably bound to interactions. Those actions provide *physical models for energy*

$$\underbrace{V_{\mathrm{pot}}\left[w_{\mathbf{1}}\right]}_{=:\, E_{\mathbf{1}}} \quad \overset{(71)}{\hookrightarrow} \quad E_{\mathrm{kin}}\left[\textcircled{i}_{\mathbf{v}_i} \Rightarrow \textcircled{i}_{\mathbf{0}}\right] \quad \overset{(75)}{\hookrightarrow} \quad V_{\mathrm{pot}}\left[\textcircled{I}_{\mathbf{x}_I} \;\Rightarrow\; \textcircled{I}_{\mathbf{x}'_I}\right] \quad . \tag{77}$$

Together with standard action $w_{\mathbf{1}}$ we provide equivalent energetic units. We presuppose the potential energy $V_{\mathrm{pot}}\left[w_{\mathbf{1}}\right] =: E_{\mathbf{1}}$ of standard action $w_{\mathbf{1}}$ *unquantified* - but exactly reproducible by energetic unit $\mathbf{1}_E\big|_{\mathbf{0}}$. The kinetic energy of generic particle $E_{\mathrm{kin}}\left[\textcircled{i}_{\mathbf{v}_i} \Rightarrow \textcircled{i}_{\mathbf{0}}\right]$ becomes measurable by an absorption action $W_{\mathrm{cal}}^{(i)}$. In return for absorbing individual particle $\textcircled{i}_{\mathbf{v}_i}$ a number $\sharp\left\{\mathbf{1}_E\big|_{\mathbf{0}}\right\}$ of equivalent energetic units is extracted from the calorimeter reservoir $\{\textcircled{1}_{\mathbf{0}}\}$. Here we employ the equipollence principle for the first time: The calorimeter extract for absorption $\mathrm{RB}\left[\textcircled{i}_{\mathbf{v}_i} \Rightarrow \textcircled{i}_{\mathbf{0}}\right] \;\sim_E\; \textcircled{i}_{\mathbf{v}_i}$ has same energy (71). We *quantify* kinetic energy of each particle $\textcircled{i}_{\mathbf{v}_i}$ by the number of equivalent energetic units in calorimeter model $W_{\mathrm{cal}}^{(i)}$.

The potential energy of - configuration transitions in a - closed system $V_{\mathrm{pot}}\left[\textcircled{I}_{\mathbf{x}_I} \;\Rightarrow\; \textcircled{I}_{\mathbf{x}'_I}\right]$ becomes measurable by model ${W_{\mathrm{cal}}^{(I)}}^{-1} * w * W_{\mathrm{cal}}^{(I)}$ for the extraction of the kinetic effect of action w from elements of system $\{\textcircled{I}_{\mathbf{x}_I}\}$. We employ the equipollence principle again:

[23] Potential energy of *configuration state* $V_{\mathrm{pot}}\left[\textcircled{I}_{\mathbf{x}_I}\right] := V_{\mathrm{pot}}\left[\textcircled{I}_{\mathbf{x}_I} \Rightarrow \textcircled{I}_{\mathrm{sep}}\right]$ is defined by means of extractable energy from transition into fixed reference configuration $\textcircled{I}_{\mathrm{sep}}$ (e.g. after separation action (88) where elements are without inner binding).

the calorimeter extract for absorbing - changed final state of motion for - all elements RB $\left[\textcircled{\scriptsize I}_{\mathbf{v}_I} \Rightarrow \textcircled{\scriptsize I}_{\mathbf{v}'_I}\right] \sim_E -\left[\textcircled{\scriptsize I}_{\mathbf{x}_I} \Rightarrow \textcircled{\scriptsize I}_{\mathbf{x}'_I}\right]$ has opposite energy than the transition of the configuration. We *quantify* potential energy - of configuration transitions - in system $\{\textcircled{\scriptsize I}_{\mathbf{x}_I}\}$ by the total number $\sharp\left\{\mathbf{1}_E\big|_{\mathbf{0}}\right\}$ of equivalent energetic units from all calorimeter measurements $W_{\text{cal}}^{(I)}$.

We begin from an unquantified but reproducible standard for unit energy $E_{\mathbf{1}} := V_{\text{pot}}[w_{\mathbf{1}}]$. By coupling (solely congruent) unit actions $w_{\mathbf{1}}$ (in a controlled way) we construct a physical model W_{cal} for absorbing kinetic energy from individual particles (71) and for separating the kinetic effect of potential energy in a system (75). We quantify the associated energy by the number $\sharp\left\{\mathbf{1}_E\big|_{\mathbf{0}}\right\}$ of equivalent energetic units $\mathbf{1}_E\big|_{\mathbf{0}}$ and its unit energy $E_{\mathbf{1}} := E\left[\mathbf{1}_E\big|_{\mathbf{0}}\right]$.

Remark 15 *By means of our physical models (76) we make the transition (77) from* <u>*units*</u> *of unquantified potential energy* $V_{\text{pot}}[w_{\mathbf{1}}]$ *over the quantification of kinetic energy* $E_{\text{kin}}[\textcircled{\scriptsize i}_{\mathbf{v}_i}]$ *to the quantification of potential energy* $V_{\text{pot}}[\textcircled{\scriptsize I}_{\mathbf{x}_I}]$.

They become measurable by the number of extractable energetic units in respective calorimeter model (see Remark 3). The whole quantification method is based on the equipollence of cause (kinetic resp. potential energy) and their - measurable - effect in our physical model.

Equipollence and conservation of energy are equivalent principles. The conservation of energy is very far from the status of an empirical law - as Schlaudt explains [23] - much more it is the basis for the quantification of 'vis viva' $E_{\text{kin}}[\textcircled{\scriptsize a}_{\mathbf{v}_a}]$, the kinetic energy of individual particles and the *condition* for measurements of potential energy $V_{\text{pot}}[\textcircled{\scriptsize I}_{\mathbf{x}_I}]$ in mechanical systems. The Equipollence Principle as the principle of equivalence of cause and effect resp. the Principle of Conservation of Energy resp. the Principle of the Impossibility of a Perpetuum Mobile have their place not in physics but in a measurement theory. Lorenzen speaks of a *measurement-theoretical a priori* [23]. Schlaudt explains: once one has accepted the conservation of energy in the sense of a measurement-theoretical a priori as the basis for quantification of energy and energy seems to be lost - then one simply did not consider a *closed system.*

4 Potential of Mechanical System

For the analysis of action w in a closed system $G_1 \cup \ldots \cup G_N$ we acknowledge: In a physical system one can never get rid of the interaction between its elements G_i. Though physicists can do the contrary, they can include additional actions into the system. We must analyze the system in the presence of both actions. Physicists can *control the process* of action w by a series of interventions. They can - temporarily - couple their calorimeter into the system $G_1 \cup \ldots \cup G_N$ to prepare the initial state of motion for individual elements and steer an undisturbed interaction: By the controlled association of brief *steering actions* $\text{RB}^{(i)}$ and consecutive segments of undisturbed action w_i physicists can successively steer the interacting system into any direction of the configuration space $\mathcal{C}$ (see figure 19).

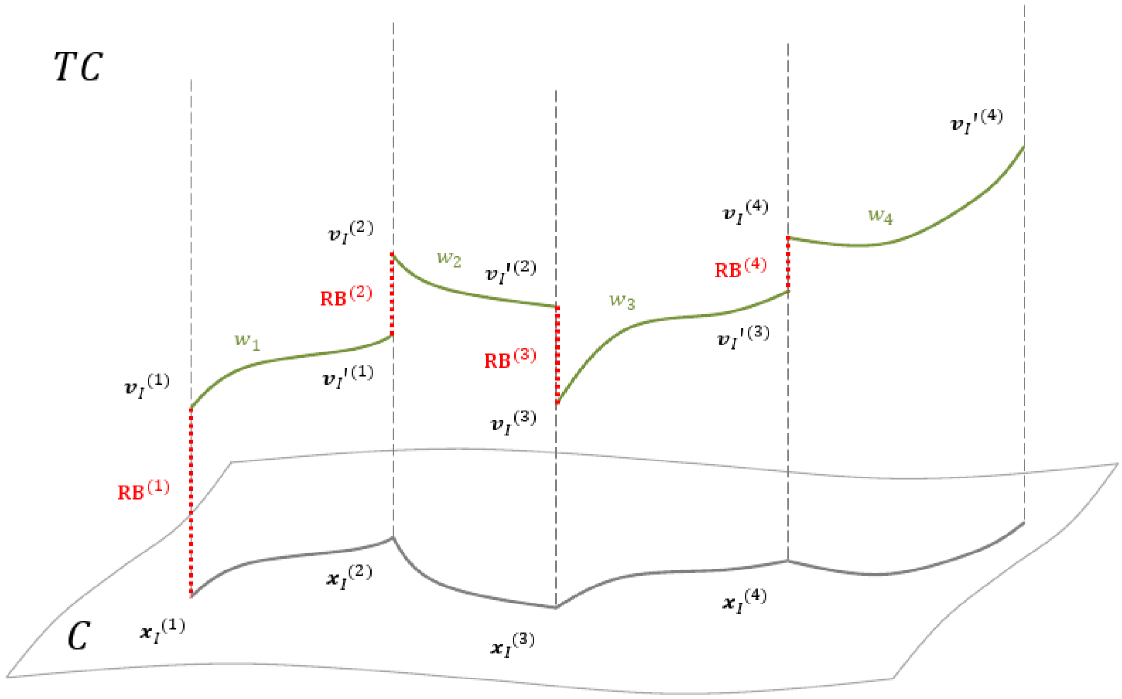

Figure 19: consecutive association of actions $\mathrm{RB}^{(i)} * w_i * \mathrm{RB}^{(i+1)} * \dots$

4.1 Steering Action

For example consider the contraction action w of a charged spring with massive bodies $\textcircled{a}, \textcircled{b}$ attached on both ends (see figure 22). The undisturbed progression of contraction action w would evolve from initially stretched configuration $(x_1, t_1) \Rightarrow (x_2, t_2) \Rightarrow (x_3, t_3)$ with increasing relative velocity towards a less stretched final configuration (x_3, t_3). Though we can also steer the contraction action w through a varied intermediate configuration $(x_1, t_1) \Rightarrow (x_2 + \delta x_2, t_2) \Rightarrow (x_3, t_3)$. At the same initially stretched configuration (x_1, t_1) a practically instantaneous calorimeter action $\mathrm{RB}^{(1)} : \textcircled{a}_{\mathbf{0}}, \textcircled{b}_{\mathbf{0}} \Rightarrow \textcircled{a}_{\mathbf{v}}, \textcircled{b}_{-\mathbf{v}}$ catapults both objects into opposite motion. Under modified initial conditions the contraction action w_1 evolves towards an even more stretched intermediate configuration $(x_2 + \delta x_2, t_2)$. Another (instantaneous) kick of the right strength $\mathrm{RB}^{(2)} : \textcircled{a}_{\mathbf{0}}, \textcircled{b}_{\mathbf{0}} \Rightarrow \textcircled{a}_{\mathbf{v}'}, \textcircled{b}_{-\mathbf{v}'}$ provides enough momentum such that the next segment of contraction action w_2 evolves to the same final configuration (x_3, t_3) - with higher velocity though. In the final steering action $\mathrm{RB}^{(3)}$ we extract the surplus kinetic energy and momentum from both elements $\textcircled{a}, \textcircled{b}$ such that the system continues evolving like contraction action w in the same undisturbed way.

As second example consider the exploration of a gravitational interaction. We illustrate the practical interplay between brief steering actions $\mathrm{RB}^{(i)}$ and consecutive segments of undisturbed gravitational actions w_i in recent GRAIL mission: NASA manufactured two satellites on Earth and launched them towards the Moon. In this physical system we can never get rid of the action e.g. by turning off the gravitational interaction, making a virtual displacement $\delta \mathbf{x}_I$ of both satellites to the Moon and turning on the interaction in the final configuration. Every change in the relative configuration $\mathbf{x}_I \Rightarrow \mathbf{x}'_I$ of Earth, both satellites, Moon etc. happens under the mutual interaction of all elements of the system.

When a physicist wants to examine an interactive system he can only intervene by a series of steering actions. At the start of a satellite mission we - temporarily - *couple* huge booster rockets against elements of the system. Two satellites are launched into initial velocity for departure from Earth. Once most fuel is burnt those boosting rockets are *decoupled* from our satellites and drop back in Earth's atmosphere. Under these new initial conditions both satellites propagate freely due to gravitational interactions from Earth to Moon. This first launching phase $\mathrm{RB}^{(1)}$ takes only a brief moment $\Delta t_{\mathrm{launch}} \ll \Delta T_{E\to M}$ when compared to the duration of the following gravitational action w_1 along the trip from Earth to Moon $\Delta T_{E\to M}$. Also the brief boost by rocket propulsion causes negligible change to the actual configuration $\Delta \mathbf{x}_I^{(\mathrm{boost})} \ll \Delta \mathbf{x}_I^{(E\to M)}$ of the system. The first steering action

$$\mathrm{RB}^{(1)}: \quad \mathbf{x}_I^{(1)}, \mathbf{v}_I = 0 \;\; \Rightarrow \;\; \mathbf{x}_I^{(1)}, \mathbf{v}_I^{(1)}$$

changes the initial velocity $\mathbf{v}_I^{(1)}$ of the satellites at practically the same configuration of the system $\mathbf{x}_I^{(1)}$ in a practically instantaneous way. Physicists control the initial conditions of the satellites for the gravitational interaction

$$w_1: \quad \mathbf{x}_I^{(1)}, \mathbf{v}_I^{(1)} \;\; \Rightarrow \;\; \mathbf{x}_I^{(2)}, \mathbf{v}'^{(1)}_I \quad .$$

Instead of remaining bound to Earth now both satellites propagate - during a long segment of free gravitational interaction - to the Moon. The (practically instantaneous) steering action $\mathrm{RB}^{(1)}$ adjusts the initial conditions $\mathbf{v}_I^{(1)}$ such that undisturbed gravitational action w_1 realizes the desired configuration transition $\mathbf{x}_I^{(1)} \Rightarrow \mathbf{x}_I^{(2)}$ from Earth to Moon. We can practically *separate* steering action $\mathrm{RB}^{(1)}$ from consecutive gravitational action w_1: During the short time when the rocket catapults the satellite into motion the effect of gravity is *comparably negligible*. Once the steering device is decoupled the effect of gravitational interaction accumulates without further external interventions (see corresponding vertical resp. horizontal paths in velocity space $T\mathcal{C}$ in figure 19).

By a consecutive sequence of temporary boosts of steering rockets $\mathrm{RB}^{(i)}$ followed by the next segment of free gravitational action w_i the physicist can navigate to any configuration of both satellites around the Moon. Without restricting generality we assume that every boost of the steering rockets, i.e. every steering intervention requires a time which is negligibly short when compared to the next segment of free gravitational propagation of both satellites. Each (practically instantaneous) steering action $\mathrm{RB}^{(i)}$ prepares the initial conditions (state of motion of both satellites) such that the consecutive undisturbed gravitational action w_i in the system of satellites and Moon evolves $\mathbf{x}_I^{(i)} \Rightarrow \mathbf{x}_I^{(i+1)}$ in a controlled way: NASA engineers set up both satellites in a standard formation around Moon. By analyzing the tidal effects on the satellites orbits in the consecutive gravitational action w_i the GRAIL mission measures the gravitational potential and eventually maps out the distribution of gravitating matter on the Moon.

The physicist *controls the process* of the interaction by repetitive steering actions $\mathrm{RB}^{(i)}$. At practically fixed configuration $\mathbf{x}_I^{(i)}$ of the system each intervention modifies the state of motion of its elements

$$\mathrm{RB}^{(i)}: \quad \mathbf{x}_I^{(i)}, \mathbf{v}_I'^{(i-1)} \;\Rightarrow\; \mathbf{x}_I^{(i)}, \mathbf{v}_I^{(i)} \tag{78}$$

in a practically instantaneous way. We can realize those interventions by our physical model for the (reversible) absorption action in a calorimeter W_{cal}: One can extract kinetic effects of gravitational actions from elements of the system {3.4.2} but also the reverse. *Engine*ers can couple additional kinetic effects (of an external calorimeter reservoir) against individual elements of the system. When coupled into the system - each steering action only effects the state of motion of the respective element at practically the same location. We regard each steering action $\mathrm{RB}^{(i)}$ as instantaneous *'kick'*. We determine those *collision actions* only with regard to changes in initial and final state of motion by complete abstraction from the inner dynamics of the process. The corresponding change in the (collective) state of motion $\mathbf{v}_I'^{(i-1)} \Rightarrow \mathbf{v}_I^{(i)}$ happens in such a comparably short moment that we can represent the kinetic effect of each steering 'kick' $\mathrm{RB}^{(i)}$ as a *vertical lift* in the velocity space $T_{\mathbf{x}_I^{(i)}}\mathcal{C}$ over practically the same configuration $\mathbf{x}_I^{(i)}$ of the system (see vertical fibres in figure 19).

Then under new initial conditions $\mathbf{v}_I^{(i)}$ intrinsic action w_i evolves free, i.e. without steering interventions from external physicists

$$w_i: \quad \mathbf{x}_I^{(i)}, \mathbf{v}_I^{(i)} \;\Rightarrow\; \mathbf{x}_I^{(i+1)}, \mathbf{v}_I'^{(i)} \quad .$$

to a new configuration $\mathbf{x}_I^{(i+1)}$ (see horizontal path in velocity space $T\mathcal{C}$ in figure 19). There the next steering action $\mathrm{RB}^{(i+1)}$ *prepares* new initial conditions for the consecutive part of free action w_{i+1} etc.

Remark 16 *Our <u>construction method</u> for the physical specification of interacting systems solely refers to two types of physical processes:*

- *intrinsic action w_i in an isolated system*
- *extrinsic (practically instantaneous) steering action* $\mathrm{RB}^{(i)}$ *and*
- *consecutive association of both actions* $\mathrm{RB}^{(1)} * w_1 * \mathrm{RB}^{(2)} * w_2 * \ldots$ *in a controlled way.*

Our physical examination of interacting system $G_1 \cup \ldots \cup G_N$ involves two different actions: intrinsic action of the system w and steering actions $\mathrm{RB}^{(i)}$ when external calorimeter $\{①_{\mathbf{0}}\}$ is coupled against individual elements G_i of system $\{G_I\} := G_1 \cup \ldots \cup G_N$. We generate each steering kick $W_{\mathrm{cal}}^{(i)}$ by coupling congruent unit actions $w_{\mathbf{1}}$ in an organized way (see Remark 9). Eventually engineers steer the process of intrinsic (e.g. gravitational) action w by congruent unit actions $w_{\mathbf{1}}$ (which are extrinsic to the system).

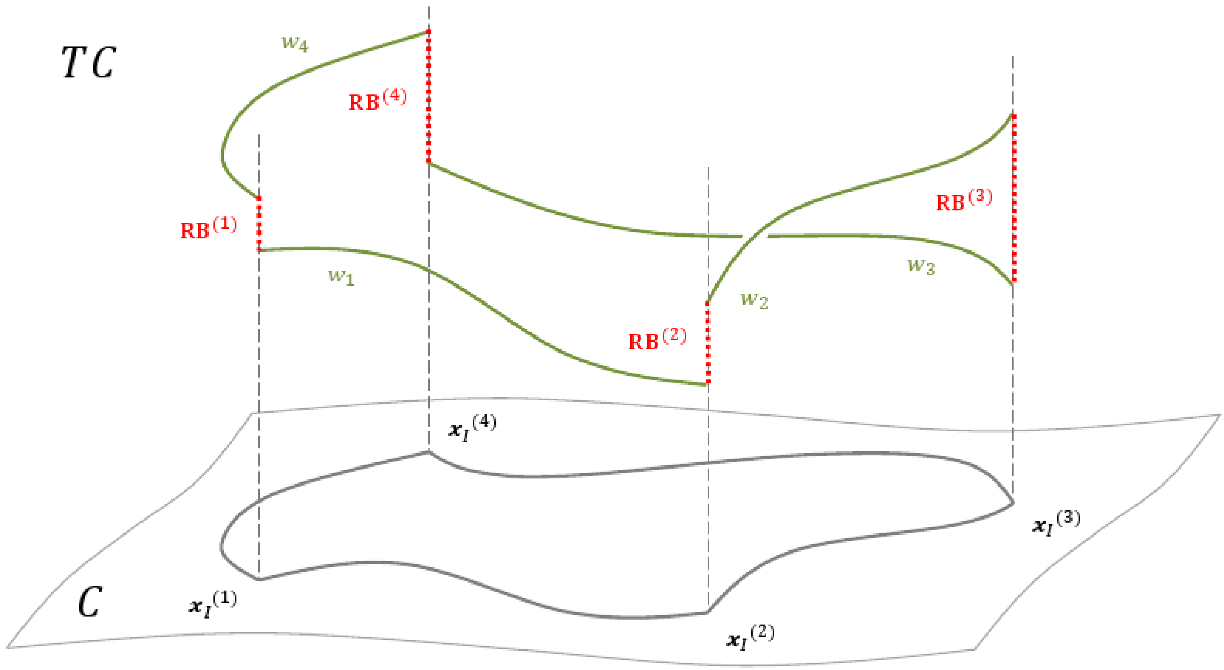

Figure 20: controlled circular process

The approximate *separability* between intrinsic action w of the system and extrinsic unit action $\mathbf{w_1}$ of the measurement device allows for - reproducible - measurement operations. The extent to which we can *approximate* unit action $\mathbf{w_1}$ (resp. steering action $\mathrm{RB}^{(i)}$ by calorimeter model) as instantaneous steering 'kick' in comparison with intrinsic (e.g. gravitational) action w allows for the implementation of following steering operations for measurements of the potential of conservative systems (depicted as vertical and horizontal transitions in velocity space $T\mathcal{C}$ in figure 19).

Lemma 7 *By consecutive association of intrinsic actions w_i and extrinsic steering actions* $\mathrm{RB}^{(i)}$ *we can trace out any (possible) path $\gamma \subset \mathcal{C}$ in configuration space $\mathcal{C}$ of the system:*

1. *circular process $\gamma_1 * \gamma_2 * \ldots * \gamma_n$ which begins and ends $\gamma_1(0) = \gamma_n(t_n) = \mathbf{x}_I^{(1)}$ at same configuration $\mathbf{x}_I^{(1)}$ of the system (see figure 20)*
2. *reversion of configuration path $\gamma * \gamma^{-1}$ (see figure 18)*
3. *complete dynamical fixation of the configuration $\mathbf{x}_I \Rightarrow \mathbf{x}_I$*
4. *partial fixation of configuration $\mathbf{x}_I \Rightarrow \mathbf{x}_I + \delta\mathbf{x}_n$ of the system except free variations for individual element $\mathbf{x}_n \in \{\mathbf{x}_I\} := \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$.*

Proof: Consider a fixed series of configurations of the system $\mathbf{x}_I^{(1)}, \mathbf{x}_I^{(2)}, \ldots, \mathbf{x}_I^{(n)} \in \mathcal{C}$ in configuration space $\mathcal{C}$. Let corresponding series of isolated actions $w_i : \mathbf{x}_I^{(i)}, \mathbf{v}_I^{(i)} \Rightarrow \mathbf{x}_I^{(i+1)}, \mathbf{v'}_I^{(i)}$ with $i = 1, \ldots, n$ realize the transition between consecutive configurations where final free action w_n leads back to starting configuration $\mathbf{x}_I^{(n+1)} \equiv \mathbf{x}_I^{(1)}$ so that the circle in configuration space $\mathcal{C}$ closes. We modify the final state of motion $\mathbf{v'}_I^{(i-1)}$ from previous free action w_{i-1} by suitable steering 'kicks' from the calorimeter reservoir $\mathrm{RB}^{(i)} : \mathbf{x}_I^{(i)}, \mathbf{v'}_I^{(i-1)} \Rightarrow \mathbf{x}_I^{(i)}, \mathbf{v}_I^{(i)}$. We prepare the initial conditions $\mathbf{v}_I^{(i)}$ (i.e. for all elements of the system we adjust the individual state of motion) at practically fixed configuration $\mathbf{x}_I^{(i)}$ for following free action w_i. Consecutive association

$$\mathrm{RB}^{(1)} * w_1 * \mathrm{RB}^{(2)} * w_2 * \ldots * \mathrm{RB}^{(n)} * w_n$$

of steering actions $\mathrm{RB}^{(i)}$ and undisturbed actions w_i provides a circular process through given configurations $\mathbf{x}_I^{(1)}, \mathbf{x}_I^{(2)}, \ldots, \mathbf{x}_I^{(n)} \in \mathcal{C}$ of the system (see figure 20). Physicists *steer* through circular process $\gamma : (\mathbf{x}_I^{(1)}, t_1) \Rightarrow (\mathbf{x}_I^{(2)}, t_2) \Rightarrow \ldots \Rightarrow (\mathbf{x}_I^{(n)}, t_n)$ back to same configuration point $\mathbf{x}_I^{(1)} \in \mathcal{C}$ - generically in arbitrary duration t_n.[24]

As special case consider a circular process between just two fixed configurations $\mathbf{x}_I^{(1)}, \mathbf{x}_I^{(2)} \in \mathcal{C}$ of the system. Let free actions

$$\begin{array}{llcll} w: & \mathbf{x}_I^{(1)}, \mathbf{v}_I^{(1)} & \Rightarrow & \mathbf{x}_I^{(2)}, \mathbf{v}_I^{(2)} & \quad \text{and} \\ \tilde{w}: & \mathbf{x}_I^{(2)}, \tilde{\mathbf{v}}_I^{(2)} & \Rightarrow & \mathbf{x}_I^{(1)}, \tilde{\mathbf{v}}_I^{(1)} & \end{array}$$

realize the (reversible) transition $\mathbf{x}_I^{(1)} \Leftrightarrow \mathbf{x}_I^{(2)}$ between the same configuration of the system under respective initial conditions $\mathbf{v}_I^{(1)}$ and $\tilde{\mathbf{v}}_I^{(2)}$. In consecutive association with suitable steering actions

$$\mathrm{RB}^{(i)} : \quad \mathbf{x}_I^{(i)}, \tilde{\mathbf{v}}_I^{(i)} \;\Rightarrow\; \mathbf{x}_I^{(i)}, \mathbf{v}_I^{(i)} \qquad i = 1, 2$$

at the initial $\mathbf{x}_I^{(1)}$ resp. final $\mathbf{x}_I^{(2)}$ configuration of the system we can construct a circular process

$$\mathrm{RB}^{(1)} * w * \left(-\mathrm{RB}^{(2)}\right) * \tilde{w}$$

which periodically oscillates between two given configurations $\mathbf{x}_I^{(1)}, \mathbf{x}_I^{(2)} \in \mathcal{C}$ of the system (see upper square in figure 18). Both free actions w and $\tilde{w}$ transfer - under respective initial conditions $\mathbf{v}_I^{(1)}$ and $\tilde{\mathbf{v}}_I^{(2)}$ - through the same configurations $\mathbf{x}_I^{(1)}, \mathbf{x}_I^{(2)}$ of the system in different durations $t[w] \neq t[\tilde{w}]$.

As special case fix two arbitrarily close configurations $\mathbf{x}_I, \mathbf{x}_I^{(\epsilon)} \in \mathcal{C}$ of the system. Let reversible interaction of motion

$$\begin{array}{llcll} w_\epsilon : & \mathbf{x}_I, \mathbf{v}_I & \Rightarrow & \mathbf{x}_I^{(\epsilon)}, \mathbf{v}_I^{(\epsilon)} & \quad \text{and} \\ \tilde{w}_\epsilon : & \mathbf{x}_I^{(\epsilon)}, \tilde{\mathbf{v}}_I^{(\epsilon)} & \Rightarrow & \mathbf{x}_I, \tilde{\mathbf{v}}_I & \end{array} \qquad (79)$$

[24] Every variation from an undisturbed action w in fixed duration $t[w]$ is associated with extra 'steering effort' (see Hamilton Principle {6}).

realize the transition $\mathbf{x}_I \Leftrightarrow \mathbf{x}_I^{(\epsilon)}$ between both neighboring configurations. By means of (comparably instantaneous) steering actions

$$\begin{array}{rcl} \mathrm{RB} &:& \mathbf{x}_I, \tilde{\mathbf{v}}_I \;\;\Rightarrow\;\; \mathbf{x}_I, \mathbf{v}_I \qquad \text{and} \\ \mathrm{RB}^{(\epsilon)} &:& \mathbf{x}_I^{(\epsilon)}, \tilde{\mathbf{v}}_I^{(\epsilon)} \;\;\Rightarrow\;\; \mathbf{x}_I^{(\epsilon)}, \mathbf{v}_I^{(\epsilon)} \end{array}$$

at the initial $\mathbf{x}_I$ resp. final $\mathbf{x}_I^{(\epsilon)}$ configuration of the system we prepare the actual state of motion of its elements. Consecutive association with undisturbed action (79)

$$W^{(\epsilon)} := \;\; \mathrm{RB} * w_\epsilon * \left(-\mathrm{RB}^{(\epsilon)}\right) * \tilde{w}_\epsilon$$

provides circular process $W^{(\epsilon)}$ which periodically oscillates around fixed configuration $\mathbf{x}_I \in \mathcal{C}$ of the system. Elementary *fixation oscillation* $W^{(\epsilon)}$ between two arbitrarily close configurations $\mathbf{x}_I, \mathbf{x}_I^{(\epsilon)} \in \mathcal{C}$ takes duration $t_\epsilon := t[W^{(\epsilon)}]$. By suitable refinement of free action w_ϵ (79) repetition of $N_\epsilon := \frac{T}{t_\epsilon}$ fixation oscillations in the system

$$W^{(\epsilon)} * \ldots * W^{(\epsilon)}$$

approximates the fixation of all elements in given configuration $(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathbf{x}_I \in \mathcal{C}$ for duration T to any adjustable precision.

Similarly for reversible interaction of motion w_ϵ (79) we provide partial steering actions

$$\mathrm{RB}_{I\backslash n} : \quad \mathbf{x}_I, \tilde{\mathbf{v}}_{I\backslash n} \;\;\Rightarrow\;\; \mathbf{x}_I, \mathbf{v}_{I\backslash n}$$

which prepare the state of motion for all elements $G_i \in G_1 \cup \ldots \cup G_N$ of the system except for element G_n. In association with undisturbed action w_ϵ (79)

$$W_{I\backslash n} := \;\; \mathrm{RB}_{I\backslash n} * w_\epsilon * \left(-\mathrm{RB}_{I\backslash n}^{(\epsilon)}\right) * \tilde{w}_\epsilon \tag{80}$$

provides a *partial fixation* process which absorbs the effect of interaction w on all elements of the system except for one element G_n. Repetitive partial fixation oscillations

$$W_{I\backslash n} * \ldots * W_{I\backslash n}$$

separate the *partial effect* of action w on individual element G_n when all other elements $\{G_1, \ldots, \overline{G_n}, \ldots, G_N\}$ of the system are (kept) in fixation.

□

Remark 17 *Intrinsic action w and extrinsic steering actions* $\mathrm{RB}^{(i)}$ *(resp. unit action $w_\mathbf{1}$) provide the physical model for the differentiated analysis of mechanical systems.*

We justify physical quantities and corresponding mathematical formulation from principles for measuring (pre-theoretic notions energy and impulse of) actions. We derive the formalism of Classical Mechanics under specified assumptions. The *limitation* of quantitative equations stem from the admissibility of empirical approximations underlying our material models for basic measurements. E.g. in the case of Gravity the spatiotemporal domain of gravitational interactions and the domain of calorimetric resp. steering actions justify a practical separation to arbitrary precision. By comparably instantaneous steering and measurement actions at practically the same configuration of the system we can provide a reproducible quantification of gravitational interactions.

4.2 Potential Field

We can steer intrinsic action w along any path $\gamma \subset \mathcal{C}$ in configuration space of the system. By calorimetric steering actions $\mathrm{RB}^{(i)}$ we adjust the initial conditions and extract the kinetic effect of configuration changes in *closed system* $\{G_I\}$. We distinguish inner elements and elements of the external calorimeter reservoir $\{①_{\mathbf{0}}\}$. They remain neutral with regard to intrinsic action w unless (a collective of) physicists couples them in absorption process W_{cal}.[25] Our absorption model $\mathrm{RB}^{-1}\big|_{\mathbf{x}_I} * w * \mathrm{RB}'\big|_{\mathbf{x}'_I}$ extracts the kinetic effect (73) of intrinsic action w from elements $G_i \in \{G_I\}$ of the system onto a *neutral* calorimeter reservoir $\{①_{\mathbf{0}}\}$.[26] How this *potentially extractable* energy and momentum from intrinsic action w depends on steering actions we specify below.

Calorimeter extract (73) provides reproducible quantification of potential energy $V_{\mathrm{pot}}[w]$ of configuration transition $\gamma : \mathbf{x}_I \Rightarrow \mathbf{x}'_I$. For reversible actions

$$V_{\mathrm{pot}}[\gamma] \quad := \quad V_{\mathrm{pot}}[w] \;\; / \;\; \mathrm{mod}\; w \tag{81}$$

it is independent of equivalent actions $w \sim_\gamma \tilde{w}$ which evolve - under initial conditions $\mathbf{v}_I$ resp. $\tilde{\mathbf{v}}_I$ - between same configurations $\gamma : \mathbf{x}_I \Rightarrow \mathbf{x}'_I$ of the system (see upper square in figure 18). By the principle of impossibility of a Perpetuum Mobile the total calorimeter extract of a circular process must vanish. Potential energy $V_{\mathrm{pot}}[\gamma]$ for same configuration transition γ is independent from the velocity $\mathbf{v}_I \neq \tilde{\mathbf{v}}_I$ resp. the duration $t[w] \neq t[\tilde{w}]$ in which intrinsic action w resp. $\tilde{w}$ evolves.

Similarly the potential energy $V_{\mathrm{pot}}[\gamma_1 * \ldots * \gamma_n]$ along a steered path $\gamma_1 * \ldots * \gamma_n \subset \mathcal{C}$ in configuration space of a closed system

$$V_{\mathrm{pot}}[\gamma_1 * \ldots * \gamma_n] \quad := \quad V_{\mathrm{pot}}\left[\mathrm{RB}^{(1)} * w_1 * \mathrm{RB}^{(2)} * w_2 * \ldots\right] \;\; / \;\; \mathrm{mod}\; \{w_i\}$$

[25] Physicists utilize congruent unit actions $w_{\mathbf{1}}$ as (dynamical) measurement devices. Each transfers changes in state of motion of elements G_i by suitable collision actions onto external calorimeter elements ①.

[26] Such neutral external calorimeter reservoir can be realized e.g. for the quantification of electromagnetic actions w_{EM}. In the case of gravitational actions w_{grav} we cannot assume the existence of one global external calorimeter reservoir for extended configurations of a gravitating system $G_1 \cup \ldots \cup G_N$. We can only provide a physical model for calorimetric measurements $W_{\mathrm{cal}}\big|_{\mathcal{U}_i}$ in a local neighborhood of each element G_i. In the practice of intrinsic gravitational measurements we must examine the physical connection between calorimetric measurements at adjacent locations $W_{\mathrm{cal}}\big|_{\mathcal{U}_i}$ and $W_{\mathrm{cal}}\big|_{\mathcal{U}_j}$.

is independent from equivalent actions w_i along configuration transitions $\gamma_i : \mathbf{x}_I^{(i)} \Rightarrow \mathbf{x}_I^{(i+1)}$ and corresponding steering actions $\mathrm{RB}^{(i)} : \mathbf{v'}_I^{(i-1)} \Rightarrow \mathbf{v}_I^{(i)}$ for matching their initial conditions. We *quantify* steering actions $\mathrm{RB}^{(i)}$ by calorimeter measurements {3.1.5} and intrinsic actions w_i by Equipollence Principle {3.4.2}. For each undisturbed action w_i we separate the kinetic effect (72) of potential energy from all elements of system $\{G_I\}$.

Definition 10 *In a* <u>*conservative*</u> *system* $\{G_I\}$ *the total kinetic effect of intrinsic action* w *on all elements vanishes for every circular process back to same initial configuration* $\mathbf{x}_I^{(1)} \in \mathcal{C}$.

If intrinsic action w is conservative we make the transition from well-defined potential energy $V_{\mathrm{pot}}[\gamma]$ along individual path $\gamma \subset \mathcal{C}$ in configuration space to a well-defined potential energy between any two configurations $\mathbf{x}_I \Rightarrow \mathbf{x}'_I \subset \mathcal{C}$

$$V_{\mathrm{pot}}[\mathbf{x}_I \Rightarrow \mathbf{x}'_I] \quad := \quad V_{\mathrm{pot}}[\gamma] \;\; / \; \bmod \gamma \tag{82}$$

independently from equivalent paths $\gamma \sim \tilde{\gamma}$ connecting same configurations of the system $\gamma * \tilde{\gamma}^{-1} : \mathbf{x}_I \Rightarrow \mathbf{x}'_I \Rightarrow \mathbf{x}_I$ in a circular way in configuration space $\mathcal{C}$.

Lemma 8 *For a conservative system* $\{G_I\}$ *the potential energy associated with configuration transitions in steered action* $\mathrm{RB}^{(1)} * w_1 * \ldots * \mathrm{RB}^{(n)} * w_n$

$$V_{\mathrm{pot}}[\mathbf{x}_I \Rightarrow \mathbf{x}'_I] \;\; = \;\; \sum_{i=1}^{n} V_{\mathrm{pot}}[w_i] \tag{83}$$

does not depend from steering path $\gamma_1 * \ldots * \gamma_n : \mathbf{x}_I \Rightarrow \mathbf{x}'_I$ *connecting initial and final configuration and from corresponding velocities* $\mathbf{v}_I[w_i]$ *resp. duration* $t[w_1 * \ldots * w_n]$ *in which intrinsic action* w *evolves.*

Proof: We determine the potential energy $V_{\mathrm{pot}}[\mathbf{x}_I \Rightarrow \mathbf{x}'_I]$ which is associated with a (steered) configuration transition by extracting the kinetic effect (73) from the corresponding process $W := \mathrm{RB}^{(1)} * w_1 * \ldots * \mathrm{RB}^{(n)} * w_n$ of steering actions $\mathrm{RB}^{(i)}$ (78) and segments of undisturbed intrinsic action w_i

$$\begin{aligned}
&\mathrm{RB}^{-1}\left[\mathbf{v}_I^{(0)}\right] * \left\{\mathrm{RB}^{(1)} * w_1 * \ldots * \mathrm{RB}^{(n)} * w_n\right\} * \mathrm{RB}\left[\mathbf{v'}_I^{(n)}\right] \\
&= \mathrm{RB}^{-1}\left[\mathbf{v}_I^{(0)}\right] * \Bigg\{ \underbrace{\mathrm{RB}\left[\mathbf{v}_I^{(0)}\right] * \mathrm{RB}^{-1}\left[\mathbf{v}_I^{(1)}\right]}_{\overset{(78)}{=}\mathrm{RB}^{(1)}} * w_1 * \underbrace{\mathrm{RB}\left[\mathbf{v'}_I^{(1)}\right] * \mathrm{RB}^{-1}\left[\mathbf{v'}_I^{(1)}\right]}_{=\mathrm{Id}} \\
&\qquad * \underbrace{\mathrm{RB}\left[\mathbf{v'}_I^{(1)}\right] * \mathrm{RB}^{-1}\left[\mathbf{v}_I^{(2)}\right]}_{\overset{(78)}{=}\mathrm{RB}^{(2)}} * w_2 * \underbrace{\mathrm{RB}\left[\mathbf{v'}_I^{(2)}\right] * \mathrm{RB}^{-1}\left[\mathbf{v'}_I^{(2)}\right]}_{=\mathrm{Id}} \\
&\qquad\qquad * \cdots * \mathrm{RB}^{-1}\left[\mathbf{v}_I^{(n)}\right] * w_n \Bigg\} * \mathrm{RB}\left[\mathbf{v'}_I^{(n)}\right] \\
&= \textstyle\sum_{i=1}^{n} \mathrm{RB}^{-1}\left[\mathbf{v}_I^{(i)}\right] * w_i * \mathrm{RB}\left[\mathbf{v'}_I^{(i)}\right]
\end{aligned} \tag{84}$$

where $\mathrm{RB}^{-1}\left[\mathbf{v}_I^{(0)}\right]$ provides the initial condition resp. 'starting kick' for steering process W and $\mathrm{RB}\left[\mathbf{v}_I'^{(n)}\right]$ absorbs the kinetic effect in the final configuration $\mathbf{x}_I'$. For the corresponding potential energy we obtain

$$V_{\mathrm{pot}}\left[\mathbf{x}_I \Rightarrow \mathbf{x}_I'\right] \overset{(73)(84)}{=} \sum_{i=1}^{n} V_{\mathrm{pot}}\left[w_i\right] \quad .$$

□

For completion - when we steer a conservative system along a circular process

$$\mathrm{RB}\left[\mathrm{RB}^{(1)} * w_1 * \ldots * \mathrm{RB}^{(n)} * w_n\right] = \sum_{i=1}^{n} \mathrm{RB}^{(i)} + \underbrace{\sum_{i=1}^{n} \mathrm{RB}\left[w_i\right]}_{\overset{!}{=}\,0} \overset{(\mathrm{Def.}\,10)}{=} 0 \tag{85}$$

the combined extract resp. expense from all steering actions $\mathrm{RB}^{(i)}$ vanishes.

For simplicity consider a two-partite bound system $ⓐ \cup ⓑ$. We determine potentially extractable momentum from intrinsic action $w : \mathbf{x}_I, \mathbf{v}_I \Rightarrow \mathbf{x}_I', \mathbf{v}_I'$ in system $ⓐ \cup ⓑ$ by absorptions against neutral elements of an external calorimeter reservoir $\{①_{\mathbf{0}}\}$. Each steering act RB in absorption model $\mathrm{RB}^{-1}\big|_{\mathbf{x}_I} * w * \mathrm{RB}'\big|_{\mathbf{x}_I'}$ extracts momentum from decelerating (two-partite) system $\mathbf{p}\left[ⓐ \cup ⓑ_{\mathbf{v}_I} \Rightarrow ⓐ \cup ⓑ_{\mathbf{0}}\right]\big|_{\mathbf{x}_I}$ at fixed initial $\mathbf{x}_I$ resp. final configuration $\mathbf{x}_I'$.[27] The momentum from both measurements is extracted onto the same external calorimeter reservoir. There we examine the momentum balance for steering configuration transitions $\mathbf{x}_I \Rightarrow \mathbf{x}_I'$ of system $ⓐ \cup ⓑ$.

We assume that intrinsic actions w as well as external steering resp. measurement actions $w_{\mathbf{1}}$ and W_{cal} satisfy compatibility conditions:

- Superposition principle
- equivalence of intrinsic actions for boosted systems
- intrinsic evolution determined by simultaneous initial conditions $\mathbf{x}_I$, $\mathbf{v}_I$

(to any practically sufficient precision). By the Superposition principle external steering actions effect individual elements in bound system $ⓐ \cup ⓑ$ in the same way as they would effect isolated elements $ⓐ$, $ⓑ$.[28] While steering actions couple against *individual elements*

[27] In pre-theoretic Definition 2 we introduce momentum by the impulse behavior of an object in a collision action under the condition that the colliding objects remain preserved.

[28] We assume that the effects of intrinsic action w and external steering actions $w_{\mathbf{1}}$ are simply superposed. Effects of steering action $w_{\mathbf{1}}$ against elements of system $ⓐ \cup ⓑ$ and against free elements are practically *indistinguishable*. Individual element $ⓐ$ has same inertial behavior $m_a^{(\mathrm{bound})} \overset{!}{=} m_a^{(\mathrm{sep})}$ in a bound state and in separation. The effect of controlled measurement actions $W_{\mathrm{cal}} := w_{\mathbf{1}} * \ldots * w_{\mathbf{1}}$ on individual elements of bound system $ⓐ \cup ⓑ$ and resulting physical quantities of kinetic energy and momentum (59) satisfy same quantitative equations (60) as determined from the absorption action for isolated elements in Theorem 2.

of the system they do not directly effect the *cause* $\mathcal{U}$ of intrinsic action w (see Remark 14). We examine what extractable effect source of energy $\mathcal{U}$ provides.[29] Is energetic source $\mathcal{U}$ also a source of extractable momentum?

Lemma 9 *In Galilei Kinematics no momentum is extractable from separation action* w_{sep} *(of individual elements from their bound state) and from intrinsic action* w *(configuration transitions in the system)*

$$\mathbf{p}\left[\textcircled{a} \cup \textcircled{b}_{\mathbf{x}_I,\mathbf{0}} \Rightarrow \textcircled{a}_{\mathbf{0}}, \textcircled{b}_{\mathbf{0}}\right] \;=\; 0 \tag{86}$$

$$\mathbf{p}\left[\textcircled{a} \cup \textcircled{b}_{\mathbf{x}_I,\mathbf{0}} \Rightarrow \textcircled{a} \cup \textcircled{b}_{\mathbf{x}'_I,\mathbf{0}}\right] \;=\; 0 \;. \tag{87}$$

Proof: Let reversible separation action

$$w_{\text{sep}}: \quad \textcircled{a} \cup \textcircled{b}_{\mathbf{x}_I,\mathbf{0}} \;\Rightarrow\; \textcircled{a}_{\mathbf{0}}\,,\; \textcircled{b}_{\mathbf{0}}\,,\; \text{RB}_{\text{sep}} \tag{88}$$

isolate elements $\textcircled{a}$ and $\textcircled{b}$ from one another in their bound state by a series of steering actions (see figure 21). The calorimeter extract for steering the separation action, i.e. the reservoir balance for separation $\text{RB}_{\text{sep}} := \text{RB}\left[\textcircled{a} \cup \textcircled{b}_{\mathbf{x}_I,\mathbf{0}} \Rightarrow \textcircled{a}_{\mathbf{0}}, \textcircled{b}_{\mathbf{0}}\right]$ is stored in the external calorimeter reservoir $\{\textcircled{1}_{\mathbf{0}}\}$. Separation action allows (reversible) *substitution* of

- system $\textcircled{a} \cup \textcircled{b}$ including intrinsic binding energy E due to the presence of both elements in a system - aka the (binding-) 'field' - by

- system $\{\textcircled{a}\,, \textcircled{b}\,, \text{RB}_{\text{sep}}\}$ with separated elements and external causes (dynamical units $\mathbf{1}_E\big|_{\mathbf{0}}, \mathbf{1}_{\mathbf{p}} \in \text{RB}_{\text{sep}}$ in the separation extract)

whose behavior under separate boost resp. steering actions is known (41), (49), (50).[30] We boost intrinsic separation action $w_{\text{sep}}^{(\mathcal{B})} \Rightarrow w_{\text{sep}}^{(\mathcal{A})}$ from $\mathcal{B}$ob's reference system to $\mathcal{A}$lice.

Let $\mathcal{A}$lice move relative to $\mathcal{B}$ob with constant (measured value for) velocity $v_{\mathcal{A}} = v_{\mathcal{A}}^{(\mathcal{B})} \cdot v_{\mathbf{1}^{(\mathcal{B})}}$. According to compatibility conditions - for $\mathcal{B}$ob boosting bound system (including elements and intrinsic sources of binding energy)

$$\text{RB}\left[\textcircled{a} \cup \textcircled{b}_{\mathbf{0}} \Rightarrow \textcircled{a} \cup \textcircled{b}_{\mathbf{v}_{\mathcal{A}}}\right]\Big|_{\mathbf{x}_I} \;=\; \text{RB}\left[\textcircled{a}_{\mathbf{0}}\,,\; \textcircled{b}_{\mathbf{0}} \Rightarrow \textcircled{a}_{\mathbf{v}_{\mathcal{A}}}\,,\; \textcircled{b}_{\mathbf{v}_{\mathcal{A}}}\right] \;\; / \text{ mod } \mathbf{x}_I \tag{89}$$

[29] Two examples for causes of an action: In intrinsic action $w_1 : \textcircled{a} \cup \textcircled{b}_{\mathbf{x}_I,\mathbf{v}_I} \Rightarrow \textcircled{a} \cup \textcircled{b}_{\mathbf{x}'_I,\mathbf{v}'_I}$ the cause of action w_1 is the presence of elements $\textcircled{a}$ and $\textcircled{b}$ in a bound (e.g. electromagnetic or gravitational) system and the change in its configuration $\mathbf{x}_I \Rightarrow \mathbf{x}'_I$. Two isolated elements $\textcircled{a}\,, \textcircled{b}$ are not present in a system. The coupling of an external source of energy $\mathcal{U}_E$ into the system $\{\textcircled{a}\,, \textcircled{b}\,, \mathcal{U}_E\}$ provides a cause of action $w_2 : \textcircled{a}_{\mathbf{0}}, \textcircled{b}_{\mathbf{0}}, \mathcal{U}_E \Rightarrow \textcircled{a}_{\mathbf{v}_a}, \textcircled{b}_{\mathbf{v}_b}, \mathcal{U}_0$ when it releases its energy $E \Rightarrow 0$.

[30] Instead of separating generic system $G_1 \cup \ldots \cup G_N\big|_{\mathbf{x}_I}$ into isolated (elementary) particles G_i also transition $\mathbf{x}_I \Rightarrow \mathbf{x}_s$ into (evtl. parts with) a standard configuration is sufficient, if bound system $G_1 \cup \ldots \cup G_N\big|_{\mathbf{x}_s}$ has unproblematic reorientation and (effective) inertial behavior $m_{\mathbf{x}_s}$ as e.g. charged and locked standard spring $\mathcal{S}_E\big|_{\mathbf{0}}$ of unit action $w_{\mathbf{1}}$ in figure 5.

requires same steering effort as the boost of separated elements - independently from the configuration of the system $\mathbf{x}_I$.[31] $\mathcal{B}$ob can substitute direct boost $\mathrm{RB}^{(B)}_{\mathrm{dir}}$ of the bound system

$$\begin{array}{ccccc} w^{(A)^{-1}}_{\mathrm{sep}} : & \underbrace{\textcircled{a}_{\mathbf{0}} \cup \textcircled{b}_{\mathbf{0}}\big|_{\mathbf{x}^{(\mathcal{A})}_I}}_{\stackrel{!}{\equiv} \textcircled{a}_{\mathbf{v}_{\mathcal{A}}} \cup \textcircled{b}_{\mathbf{v}_{\mathcal{A}}}\big|_{\mathbf{x}^{(\mathcal{B})}_I}} & \Longleftarrow & \textcircled{a}_{\mathbf{0}}\,,\ \textcircled{b}_{\mathbf{0}}\,,\ \mathrm{RB}^{(\mathcal{A})}_{\mathrm{sep}} & \\ \mathrm{RB}^{(B)}_{\mathrm{dir}} & \uparrow & & \Uparrow & \mathrm{RB}^{(B)}_{\mathrm{indir}} \\ w^{(B)}_{\mathrm{sep}} : & \textcircled{a}_{\mathbf{0}} \cup \textcircled{b}_{\mathbf{0}}\big|_{\mathbf{x}^{(\mathcal{B})}_I} & \Longrightarrow & \textcircled{a}_{\mathbf{0}}\,,\ \textcircled{b}_{\mathbf{0}}\,,\ \mathrm{RB}^{(\mathcal{B})}_{\mathrm{sep}} & \end{array}$$

indirectly $\mathrm{RB}^{(B)}_{\mathrm{indir}}$ by boosting separate elements $\textcircled{a}$, $\textcircled{b}$ and steering ingredients $\mathrm{RB}^{(\mathcal{B})}_{\mathrm{sep}}$. $\mathcal{A}$lice reproduces $w^{(\mathcal{A})^{-1}}_{\mathrm{sep}}$ the (intrinsically equivalent) bound system $\textcircled{a}_{\mathbf{0}} \cup \textcircled{b}_{\mathbf{0}}\big|_{\mathbf{x}^{(\mathcal{A})}_I}$ with dynamical units $\mathbf{1}^{(\mathcal{A})}_E\big|_{\mathbf{0}}$, $\mathbf{1}^{(\mathcal{A})}_{\mathbf{p}}$ of her own (boosted) calorimeter reservoir.

Both - connected and separated - ways of boosting entire system $\textcircled{a}_{\mathbf{0}} \cup \textcircled{b}_{\mathbf{0}}\big|_{\mathbf{x}^{(\mathcal{B})}_I}$

$$-\mathrm{RB}^{(B)}_{\mathrm{dir}} \ * \ \left(w^{(B)}_{\mathrm{sep}} \ * \ \mathrm{RB}^{(B)}_{\mathrm{indir}} \ * \ w^{(A)^{-1}}_{\mathrm{sep}} \right)$$

form a circular process.[32] $\mathcal{B}$ob's steering effort for the direct boost (of the system)

$$\begin{aligned} \mathrm{RB}_{\mathrm{dir}}\left[\textcircled{a} \cup \textcircled{b}_{\mathbf{0}} \Rightarrow \textcircled{a} \cup \textcircled{b}_{\mathbf{v}_{\mathcal{A}}}\right] &\stackrel{!}{=} \mathrm{RB}_{\mathrm{indir}}\left[\textcircled{a}_{\mathbf{0}}\,,\ \textcircled{b}_{\mathbf{0}}\,,\ \mathrm{RB}^{(\mathcal{B})}_{\mathrm{sep}} \Rightarrow \textcircled{a}_{\mathbf{v}_{\mathcal{A}}}\,,\ \textcircled{b}_{\mathbf{v}_{\mathcal{A}}}\,,\ \mathrm{RB}^{(\mathcal{A})}_{\mathrm{sep}}\big|_{\mathbf{v}_{\mathcal{A}}}\right] \\ &= \mathrm{RB}\left[\textcircled{a}_{\mathbf{0}}\,,\ \textcircled{b}_{\mathbf{0}} \Rightarrow \textcircled{a}_{\mathbf{v}_{\mathcal{A}}}\,,\ \textcircled{b}_{\mathbf{v}_{\mathcal{A}}}\right] + \mathrm{RB}\left[\mathrm{RB}^{(\mathcal{B})}_{\mathrm{sep}} \Rightarrow \mathrm{RB}^{(\mathcal{A})}_{\mathrm{sep}}\big|_{\mathbf{v}_{\mathcal{A}}}\right] \qquad (90) \end{aligned}$$

[31] According to Superposition principle $\mathcal{B}$ob's steering effort $\mathrm{RB}\left[\textcircled{a}_{\mathbf{0}}\,,\ \textcircled{b}_{\mathbf{0}} \Rightarrow \textcircled{a}_{\mathbf{v}_{\mathcal{A}}}\,,\ \textcircled{b}_{\mathbf{v}_{\mathcal{A}}}\right]$ transfers system $\textcircled{a}_{\mathbf{0}} \cup \textcircled{b}_{\mathbf{0}}\big|_{\mathbf{x}^{(\mathcal{B})}_I} \Rightarrow \textcircled{a}_{\mathbf{v}_{\mathcal{A}}} \cup \textcircled{b}_{\mathbf{v}_{\mathcal{A}}}\big|_{\mathbf{x}^{(\mathcal{B})}_I} \stackrel{!}{\equiv} \textcircled{a}_{\mathbf{0}} \cup \textcircled{b}_{\mathbf{0}}\big|_{\mathbf{x}^{(\mathcal{A})}_I}$ into a state which - by equivalence of intrinsic actions $w^{(\mathcal{A})} \equiv w^{(\mathcal{B})}$ and their dependence on instantaneous initial conditions $\mathbf{x}^{(\mathcal{A})}_I \equiv \mathbf{x}^{(\mathcal{B})}_I$ - reproduces the *intrinsically equivalent* system with regard to the intrinsic reference frame of $\mathcal{A}$lice.

[32] Without compatibility conditions between intrinsic action w and external steering actions W_{cal} the equivalence of steering bound system $\textcircled{a} \cup \textcircled{b}$ and isolated elements $\textcircled{a}$, $\textcircled{b}$ is lost. $\mathcal{B}$ob's steering actions $\mathrm{RB}\left[\textcircled{a}_{\mathbf{0}}\,,\ \textcircled{b}_{\mathbf{0}} \Rightarrow \textcircled{a}_{\mathbf{v}_{\mathcal{A}}}\,,\ \textcircled{b}_{\mathbf{v}_{\mathcal{A}}}\right]$ on individual *elements* $\textcircled{a}_{\mathbf{0}} \cup \textcircled{b}_{\mathbf{0}}\big|_{\mathbf{x}^{(\mathcal{B})}_I} \Rightarrow \textcircled{a}_{\mathbf{v}_{\mathcal{A}}} \cup \textcircled{b}_{\mathbf{v}_{\mathcal{A}}}\big|_{\mathbf{x}^{(\mathcal{B})}_I} \not\equiv \textcircled{a}_{\mathbf{0}} \cup \textcircled{b}_{\mathbf{0}}\big|_{\mathbf{x}^{(\mathcal{A})}_I}$ are not sufficient to reproduce an intrinsically equivalent bound system for $\mathcal{A}$lice - in Poincare Kinematics. To boost an intrinsic *source* of (binding) energy requires additional steering effort.

In complete abstraction from inner dynamics in bound system $\textcircled{a} \cup \textcircled{b}$ intrinsically equivalent separation actions $w^{(B)}_{\mathrm{sep}} \ * \ \mathrm{RB}^{(B)}_{\mathrm{indir}} \ * \ w^{(A)^{-1}}_{\mathrm{sep}}$ provide a *physical connection* between e.g. boosted units of energy $\mathbf{1}^{(\mathcal{A})}_E\big|_{\mathbf{v}_{\mathcal{A}}}$ and $\mathbf{1}^{(\mathcal{B})}_E\big|_{\mathbf{v}_{\mathcal{B}}}$ (see Lemma 4). For electromagnetic actions w_{EM} binding energy in system $\textcircled{a} \cup \textcircled{b}$ is determined by the retarded relative localization of its elements. In absence of (hypothetical absolute) initial conditions intrinsic action w_{EM} is governed by the retarded Coulomb Principle. The acceleration of an extended configuration of (electromagnetically) bound system $\textcircled{a} \cup \textcircled{b}_{\mathbf{x}^{(\mathcal{A})}_I, \mathbf{0}^{(\mathcal{A})}} \Rightarrow \textcircled{a} \cup \textcircled{b}_{\mathbf{x}^{(\mathcal{B})}_I, \mathbf{0}^{(\mathcal{B})}}$ requires additional subtle steering kicks (associated to 'radiation') to reproduce an intrinsically equivalent configuration $\mathbf{x}^{(\mathcal{A})}_I \equiv \mathbf{x}^{(\mathcal{B})}_I$.

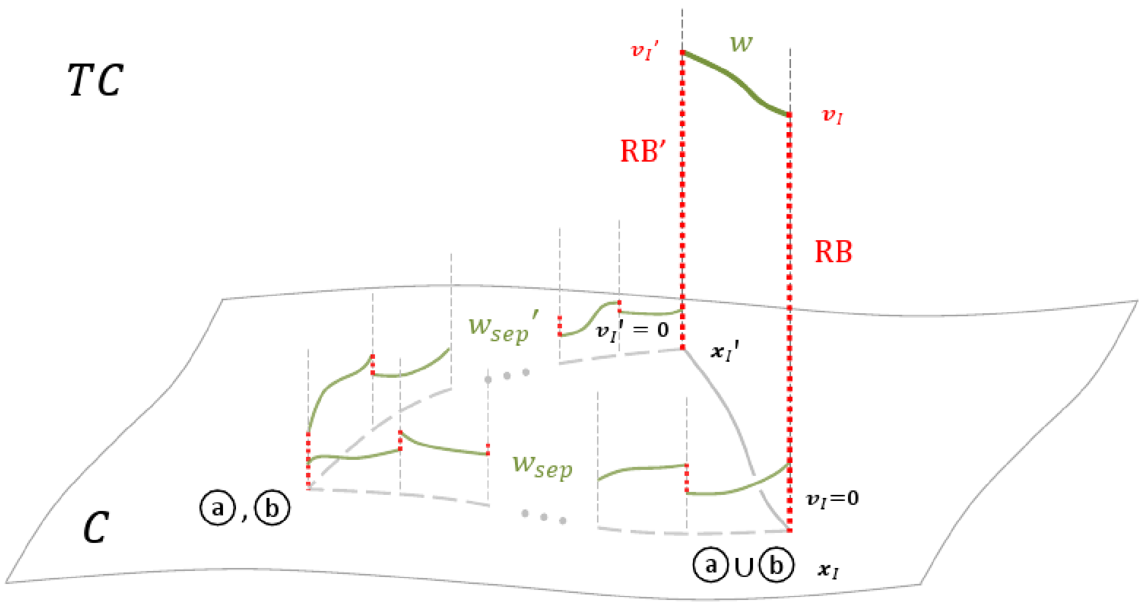

Figure 21: conservation of extractable momentum

and for steering the indirect boost (of the separated components) must be the same. $\mathcal{B}$ob's reservoir balance for boosting his steering ingredients/devices $\mathrm{RB}^{(\mathcal{B})}_{\mathrm{sep}} := k \cdot \mathbf{1}^{(\mathcal{B})}_E\big|_{\mathbf{0}}, \, l \cdot \mathbf{1}^{(\mathcal{B})}_{\mathbf{p}}$ for separation action towards $\mathcal{A}$lice must vanish

$$\mathrm{RB}^{(\mathcal{B})}\left[\mathrm{RB}^{(\mathcal{B})}_{\mathrm{sep}} \Rightarrow \mathrm{RB}^{(\mathcal{A})}_{\mathrm{sep}}\big|_{\mathbf{v}_\mathcal{A}}\right] \overset{(90)(89)}{=} 0 \ . \tag{91}$$

Hence separation action (88) can solely release units of energy (boosting $\mathbf{1}^{(\mathcal{B})}_E\big|_{\mathbf{0}}$ is for free)

$$\mathrm{RB}^{(\mathcal{B})}_{\mathrm{sep}} \ := \ k \cdot \mathbf{1}^{(\mathcal{B})}_E\big|_{\mathbf{0}}, \underbrace{l}_{\overset{!}{=}0} \cdot \mathbf{1}^{(\mathcal{B})}_{\mathbf{p}}$$

but not units of momentum. $\mathcal{B}$ob can only boost momentum units $\mathbf{1}^{(\mathcal{B})}_{\mathbf{p}} \Rightarrow \mathbf{1}^{(\mathcal{A})}_{\mathbf{p}}$ towards $\mathcal{A}$lice by consuming additional units of energy $\mathbf{1}^{(\mathcal{B})}_E\big|_{\mathbf{0}}$ (see Lemma 4) in contradiction to conservation (91). Expending intrinsic binding energy E of system ⓐ ∪ ⓑ does not release extractable momentum - which proves assertion (86).

Next consider the extractable kinetic effect from intrinsic action w in system ⓐ ∪ ⓑ

$$\left(-\mathrm{RB}\big|_{\mathbf{x}_I}\right) * w * \mathrm{RB}'\big|_{\mathbf{x}'_I} : \quad ⓐ \cup ⓑ_{\mathbf{x}_I,\mathbf{0}} \ \Rightarrow \ ⓐ \cup ⓑ_{\mathbf{x}'_I,\mathbf{0}}$$

where steering action RB prepares initial state of motion and RB′ absorbs the kinetic effect (see figure 21). Let separation actions w_{sep} and w'_{sep} prepare initial $\mathbf{x}_I$ and final configuration

$\mathbf{x}'_I$ of the system from separated elements $Ⓐ$, $Ⓑ$ by expending corresponding separation extract $\mathrm{RB}_{\mathrm{sep}}$ resp. $\mathrm{RB}'_{\mathrm{sep}}$ (88). After steering the entire system along circular process

$$W \; := \; (-\mathrm{RB}) \, * \, w \, * \, \mathrm{RB}' \, * \, w'_{\mathrm{sep}} \, * \, {w_{\mathrm{sep}}}^{-1}$$

the combined calorimeter extract $\mathrm{RB}\,[W] \stackrel{!}{=} 0$ must vanish (85). Hence potential momentum is neither extractable from the complete circular process

$$\mathbf{p}\,[\mathrm{RB}\,[W]] \; \stackrel{!}{=} \; 0 \; = \; \mathbf{p}\,[(-\mathrm{RB}) * w * \mathrm{RB}'] + \underbrace{\mathbf{p}\,\left[w'_{\mathrm{sep}}\right]}_{\stackrel{(86)}{=}\,0} + \underbrace{\mathbf{p}\,\left[{w_{\mathrm{sep}}}^{-1}\right]}_{\stackrel{(86)}{=}\,0}$$

nor from the kinetic effect of intrinsic action w - which proves assertion (87).

□

The extractable momentum from (absorbing and separating) the entire system

$$\begin{aligned} \mathbf{p}\,[Ⓐ \cup Ⓑ_{\mathbf{x}_I,\mathbf{v}_I}] \quad &:= \quad \mathbf{p}\,[Ⓐ \cup Ⓑ_{\mathbf{x}_I,\mathbf{v}_I} \Rightarrow Ⓐ \cup Ⓑ_{\mathbf{x}_I,\mathbf{0}}] \; + \; \mathbf{p}\,[Ⓐ \cup Ⓑ_{\mathbf{x}_I,\mathbf{0}} \Rightarrow Ⓐ_{\mathbf{0}}, Ⓑ_{\mathbf{0}}] \\ &\stackrel{(89)(86)}{=} \quad \mathbf{p}\,[Ⓐ_{\mathbf{v}_a}] + \mathbf{p}\,[Ⓑ_{\mathbf{v}_b}] \end{aligned} \tag{92}$$

is the total momentum of its elements. In intrinsic action w the total momentum of all elements is conserved

$$\begin{aligned} \mathbf{p}\left[\left(-\mathrm{RB}\big|_{\mathbf{x}_I}\right) * w * \mathrm{RB}'\big|_{\mathbf{x}'_I}\right] \;\; &= \;\; \mathbf{p}\left[-\mathrm{RB}\big|_{\mathbf{x}_I}\right] \; + \; \underbrace{\mathbf{p}\,[w]}_{\stackrel{!}{=}\,0} + \mathbf{p}\left[\mathrm{RB}'\big|_{\mathbf{x}'_I}\right] \\ &\stackrel{(89)}{=} \;\; -\mathbf{p}\,[Ⓐ_{\mathbf{v}_a}] - \mathbf{p}\,[Ⓑ_{\mathbf{v}_b}] + \mathbf{p}\left[Ⓐ_{\mathbf{v}'_a}\right] + \mathbf{p}\left[Ⓑ_{\mathbf{v}'_b}\right] \stackrel{(87)}{=} \; 0 \; . \end{aligned} \tag{93}$$

No (potential) momentum is extractable from the source of energy onto the elements. They have same total momentum in the initial and final state. From the presence of the system (i.e. bound state, binding field etc.) we can extract potential form of energy but no potential form of momentum. By Superposition Principle we generalize from two-partite system to generic N-body system $G_1 \cup \ldots \cup G_N$.

Corollary 4 *In Galilei Kinematics the potential effect of intrinsic action w in system $\{G_I\}$ depends solely on configuration transitions. It is determined by* <u>*potential field*</u> $V_{\mathrm{pot}}\,[\mathbf{x}_I \Rightarrow \mathbf{x}'_I]$ *(83) and conservation of* <u>*total momentum*</u> *of its elements* $\sum_{i\in I} \mathbf{p}\,[Ⓘ_{\mathbf{v}_i}] \stackrel{!}{=} \mathrm{const}$ *(93).*[33]

[33] In Poincare Kinematics potential effects are velocity dependent. Both potential energy and momentum $(E, \mathbf{p})\,\left[\mathbf{1}_E\big|_{\mathbf{v}}\right]$ is redistributed from (moving) source of action onto elements of the system [25].

5 Differentiated Analysis

We analyze the calorimetric characterization of intrinsic actions w in conservative system $G_1 \cup \ldots \cup G_N$ in a differentiated way. How does energy and momentum redistribute between the system and all *individual elements G_i throughout the spatiotemporal process*?

Our physical specification of actions is based on the extraction of congruent dynamical units $\mathrm{RB}\big|_{G_i}$ from individual elements G_i onto an external calorimeter reservoir $\{ⓘ_{\mathbf{0}}\}$.[34] We investigate *infinitesimal* segments of intrinsic action w in system ⓐ∪ⓑ. Each configuration transition $\mathbf{x}_I \Rightarrow \mathbf{x}'_I$ generates a kinetic effect

$$\mathrm{RB}\left[ⓐ \cup ⓑ_{\mathbf{x}_I,\mathbf{0}} \Rightarrow ⓐ \cup ⓑ_{\mathbf{x}'_I,\mathbf{0}}\right] \overset{(74)}{=} -\ \mathrm{RB}\left[ⓐ_{\mathbf{v}_a} \Rightarrow ⓐ_{\mathbf{v}_a+\Delta\mathbf{v}_a}\right] -\ \mathrm{RB}\left[ⓑ_{\mathbf{v}_b} \Rightarrow ⓑ_{\mathbf{v}_b+\Delta\mathbf{v}_b}\right]$$

which by Lemma 8, 9 is independent from steering path and initial condition $\mathbf{v}_I = (\mathbf{v}_a, \mathbf{v}_b)$. We analyze the redistribution of corresponding energy and momentum

$$\underbrace{(E,\mathbf{p})\left[\mathrm{RB}\left[ⓐ \cup ⓑ_{\mathbf{x}_I,\mathbf{0}} \Rightarrow ⓐ \cup ⓑ_{\mathbf{x}'_I,\mathbf{0}}\right]\right]}_{\overset{(\mathrm{Cor.}4)}{=}\left(V_{\mathrm{pot}}\left[\mathbf{x}_I\Rightarrow\mathbf{x}'_I\right],\ \mathbf{0}\ \right)} = -\ \underbrace{(E,\mathbf{p})\mathrm{RB}\left[\left[ⓐ_{\mathbf{v}_a} \Rightarrow ⓐ_{\mathbf{v}_a+\Delta\mathbf{v}_a}\right]\right]}_{=\Delta(E_{\mathrm{kin}},\mathbf{p})_a} -\ \underbrace{(E,\mathbf{p})\left[\mathrm{RB}\left[ⓑ_{\mathbf{v}_b} \Rightarrow ⓑ_{\mathbf{v}_b+\Delta\mathbf{v}_b}\right]\right]}_{=\Delta(E_{\mathrm{kin}},\mathbf{p})_b}$$

between elements ⓐ, ⓑ throughout the spatiotemporal evolution of action $w : t, \mathbf{x}_I \Rightarrow t', \mathbf{x}'_I$. Basic dynamical quantities - kinetic energy and momentum of individual element ⓘ ∈ ⓐ∪ⓑ from the system - obey quantitative equations (59) derived from the absorption action of free particles in a calorimeter (see Theorem 2)

$$\begin{aligned}(E,\mathbf{p})\left[ⓘ_{\mathbf{v}_i} \Rightarrow ⓘ_{\mathbf{v}_i+\Delta\mathbf{v}_i}\right] &= (E,\mathbf{p})\left[ⓘ_{\mathbf{v}_i+\Delta\mathbf{v}_i} \Rightarrow ⓘ_{\mathbf{0}}\right] - (E,\mathbf{p})\left[ⓘ_{\mathbf{v}_i} \Rightarrow ⓘ_{\mathbf{0}}\right] \\ &\overset{(59)}{=} \left(\left\{\frac{m_i^{(\mathcal{A})}}{2}\cdot\left(\mathbf{v}_i^{(\mathcal{A})}+\Delta\mathbf{v}_i^{(\mathcal{A})}\right)^2 - \frac{m_i^{(\mathcal{A})}}{2}\cdot\mathbf{v}_i^{(\mathcal{A})2}\right\}\cdot E_{\mathbf{1}^{(\mathcal{A})}}\ ,\ \left\{m_i^{(\mathcal{A})}\cdot\Delta\mathbf{v}_i^{(\mathcal{A})}\right\}\cdot\mathbf{p}_{\mathbf{1}^{(\mathcal{A})}}\right) \\ &= \left(\left\{m_i^{(\mathcal{A})}\cdot\mathbf{v}_i^{(\mathcal{A})}\cdot\Delta\mathbf{v}_i^{(\mathcal{A})}\right\}\cdot E_{\mathbf{1}^{(\mathcal{A})}}\ ,\ \left\{m_i^{(\mathcal{A})}\cdot\Delta\mathbf{v}_i^{(\mathcal{A})}\right\}\cdot\mathbf{p}_{\mathbf{1}^{(\mathcal{A})}}\right) \qquad (94)\end{aligned}$$

with higher order terms $\Delta\mathbf{v}_i^{(\mathcal{A})} \ll \mathbf{v}_i^{(\mathcal{A})}$ suppressed as segments of action w are infinitesimal.

Provided *basic* physical quantities length, duration [24] and energy, momentum {3} we introduce more differentiated termini suitable for analyzing continuous evolution. We define 'force' of intrinsic actions {5.1} and 'displacement work' in steered actions {5.2}. We determine properties and interrelations of these *derived* physical quantities and ultimately the equations of motion {5.3} for interactions in a conservative system.

[34] For simplicity we first investigate two-partite systems ⓐ∪ⓑ and then extend our analysis - by assuming Superposition Principle - to n-body configurations $G_1 \cup \ldots . \cup G_N$.

5.1 Force

Definition 11 *The* <u>*force*</u> $\mathbf{F}_a$ *- of intrinsic action* w *in system* $ⓐ_{\mathbf{v}_a} \cup ⓑ_{\mathbf{v}_b}\big|_{\mathbf{x}_a,\mathbf{x}_b}$ *with initial condition* $\mathbf{v}_I$ *at initial configuration* $\mathbf{x}_I$ *- against element* ⓐ *is a derived physical quantity which specifies how its momentum evolves*

$$\mathbf{F}_a^{(\mathcal{A})}\Big[w\big|_{\mathbf{x}_I,\mathbf{v}_I}\Big] \cdot \Delta t_a^{(\mathcal{A})} \quad := \quad \Delta \mathbf{p}_a^{(\mathcal{A})} \quad . \tag{95}$$

Corollary 5 *If element* ⓐ *moves - by its inertia and given initial conditions* $\mathbf{v}_a$ *- along infinitesimal way* $\Delta \mathbf{s}_a = \mathbf{v}_a \cdot \Delta t$ *the force* $\mathbf{F}_a^{(\mathcal{A})}$ *of intrinsic action* w *also determines how its kinetic energy evolves*

$$\mathbf{F}_a^{(\mathcal{A})}\Big[w\big|_{\mathbf{x}_I,\mathbf{v}_I}\Big] \cdot \Delta \mathbf{s}_a^{(\mathcal{A})} \quad = \quad \Delta E_{\mathrm{kin}\,a}^{(\mathcal{A})} \quad . \tag{96}$$

Proof: The force against element $ⓐ_{\mathbf{v}_a}$ at velocity $\mathbf{v}_a$ - in intrinsic action w with element $ⓑ_{\mathbf{v}_b}$ at velocity $\mathbf{v}_b$ - satisfies

$$\mathbf{F}_a^{(\mathcal{A})}\big|_{\mathbf{v}_a,\mathbf{v}_b} \cdot \Delta \mathbf{s}_a^{(\mathcal{A})} \quad \overset{(95)(94)}{=} \quad \frac{m_a^{(\mathcal{A})} \cdot \Delta \mathbf{v}_a^{(\mathcal{A})}}{\Delta t^{(\mathcal{A})}} \; \cdot \; \mathbf{v}_a^{(\mathcal{A})} \cdot \Delta t^{(\mathcal{A})} \quad \overset{(94)}{=} \quad \Delta E_{\mathrm{kin}\,a}^{(\mathcal{A})} \quad .$$

□

Lemma 10 *Let same intrinsic action* w *in system* $ⓐ_{\mathbf{v}_a} \cup ⓑ_{\mathbf{v}_b}\big|_{\mathbf{x}_a,\mathbf{x}_b}$ *be physically measured by boosted inertial observers* $\mathcal{A}$*lice and* $\mathcal{B}$*ob. Their respective physical quantities for force against elements* ⓐ *and* ⓑ *satisfy*

$$\begin{aligned} \mathbf{F}_a^{(\mathcal{A})} &= -\mathbf{F}_b^{(\mathcal{A})} \\ \mathbf{F}_a^{(\mathcal{A})} &= \mathbf{F}_a^{(\mathcal{B})} \quad . \end{aligned}$$

Proof: According to momentum conservation $\Delta \mathbf{p}_a + \Delta \mathbf{p}_b \overset{(93)}{=} 0$ for calorimetric measurements (in Galilei Kinematics) in two-partite system ⓐ ∪ ⓑ the forces against ⓐ resp. ⓑ

$$\mathbf{F}_a^{(\mathcal{A})} \quad := \quad \frac{\Delta \mathbf{p}_a^{(\mathcal{A})}}{\Delta t} \quad = \quad -\frac{\Delta \mathbf{p}_b^{(\mathcal{A})}}{\Delta t} \quad =: \quad -\mathbf{F}_b^{(\mathcal{A})}$$

are oriented antiparallel with same strength.

For infinitesimal segment of intrinsic action w in system ⓐ ∪ ⓑ

$$w: \quad ⓐ \cup ⓑ_{t,\,\mathbf{x}_I,\,\mathbf{v}_I} \;\Rightarrow\; ⓐ \cup ⓑ_{t'=t+\Delta t\,,\,\mathbf{x}'_I \simeq \mathbf{x}_I + \mathbf{v}_I \cdot \Delta t\,,\,\mathbf{v}'_I = \mathbf{v}_I + \Delta \mathbf{v}_I} \tag{97}$$

with initial conditions $\mathbf{v}_I = (\mathbf{v}_a, \mathbf{v}_b)$ set up at initial configuration $\mathbf{x}_I = (\mathbf{x}_a, \mathbf{x}_b)$ $\mathcal{A}$lice and $\mathcal{B}$ob specify the acquired kinetic effect

$$\Delta \mathbf{v}_I = \Delta \mathbf{v}_I^{(\mathcal{A})} \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{A})}} = \Delta \mathbf{v}_I^{(\mathcal{B})} \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}$$

after duration Δt. Let $\mathcal{A}$lice move relative to $\mathcal{B}$ob with constant velocity $v_{\mathcal{A}} = \mathbf{v}_{\mathcal{A}}^{(\mathcal{B})} \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}$. $\mathcal{B}$ob measured values of - initial $\mathbf{v}_I$ and final $\mathbf{v}'_I$ - velocity of same objects $\textcircled{a}, \textcircled{b}$ transform covariant $\mathbf{v}_I^{(\mathcal{B})} = \mathbf{v}_I^{(\mathcal{A})} + \mathbf{v}_{\mathcal{A}}^{(\mathcal{B})}$ (see Remark 8). Under the condition of Galilei Kinematics $\mathcal{A}$lice and $\mathcal{B}$ob measure same values of acceleration

$$\Delta \mathbf{v}_I^{(\mathcal{A})} \;=\; \Delta \mathbf{v}_I^{(\mathcal{B})}$$

in same duration $\Delta t^{(\mathcal{A})} \;=\; \Delta t^{(\mathcal{B})}$. Same quantitative equations (59) follow from intrinsically equivalent calorimetric measurements $\mathrm{RB}^{(\mathcal{A})}$ and $\mathrm{RB}^{(\mathcal{B})}$. Thus $\mathcal{A}$lice and $\mathcal{B}$ob must measure the same physical quantity

$$\Delta \mathbf{p}_I^{(\mathcal{A})} \;=\; \Delta \mathbf{p}_I^{(\mathcal{B})}$$

of momentum changes $\Delta \mathbf{p}_I = \Delta \mathbf{p}_I^{(\mathcal{A})} \cdot \mathbf{p}_{\mathbf{1}^{(\mathcal{A})}} = \Delta \mathbf{p}_I^{(\mathcal{B})} \cdot \mathbf{p}_{\mathbf{1}^{(\mathcal{B})}}$. Hence their intrinsic measurement values for force against element $\textcircled{a}$ are the same

$$\mathbf{F}_a^{(\mathcal{A})} \;:=\; \frac{\Delta \mathbf{p}_a^{(\mathcal{A})}}{\Delta t^{(\mathcal{A})}} \;=\; \frac{\Delta \mathbf{p}_a^{(\mathcal{B})}}{\Delta t^{(\mathcal{B})}} \;=:\; \mathbf{F}_a^{(\mathcal{B})} \;.$$

□

Theorem 4 *The force of intrinsic action w in conservative system $\textcircled{a}_{\mathbf{v}_a} \cup \textcircled{b}_{\mathbf{v}_b}\big|_{\mathbf{x}_a, \mathbf{x}_b}$ against element $\textcircled{a}$ is independent from the initial state of motion $\mathbf{v}_I = (\mathbf{v}_a, \mathbf{v}_b)$ of both elements*

$$\mathbf{F}_a^{(\mathcal{A})} \;:=\; \frac{\Delta \mathbf{p}_a^{(\mathcal{A})}}{\Delta t^{(\mathcal{A})}} \;/\; \mathrm{mod}\ \mathbf{v}_a, \mathbf{v}_b \;. \tag{98}$$

Proof: Let $\mathcal{A}$lice set up intrinsic action w in system $\textcircled{a} \cup \textcircled{b}_{\mathbf{x}_I, \mathbf{v}_I / \mathbf{w}_I}$ at same initial configuration $\mathbf{x}_I$ once with initial conditions $\mathbf{v}_I = (\mathbf{v}_a, \mathbf{v}_b)$ and once with initial conditions $\mathbf{w}_I = (\mathbf{w}_a, \mathbf{w}_b)$ (see figure 18). Without restricting generality $\mathcal{A}$lice *prepares* both initial velocities so that total momentum of the system $\mathbf{p}\left[\textcircled{a} \cup \textcircled{b}_{\mathbf{v}_I}\right] \stackrel{!}{=} 0$ resp. $\mathbf{p}\left[\textcircled{a} \cup \textcircled{b}_{\mathbf{w}_I}\right] \stackrel{!}{=} 0$ vanishes ($\mathcal{A}$lice represents so-called center of mass frame). Then in both actions w resp. $\tilde{w}$ elements $\textcircled{a}, \textcircled{b}$ run through same configuration changes in different duration $\Delta t \neq \Delta T$. We couple both actions into a circular process

$$\begin{array}{rccc}
w: & \textcircled{a} \cup \textcircled{b}_{\mathbf{x}_I, \mathbf{v}_I} & \xrightarrow{\Delta t} & \textcircled{a} \cup \textcircled{b}_{\mathbf{x}'_I \simeq \mathbf{x}_I + \mathbf{v}_I \cdot \Delta t\,,\, \mathbf{v}'_I = \mathbf{v}_I + \Delta \mathbf{v}_I} \\
 & {\scriptstyle \mathrm{RB}[\mathbf{v}_I \Rightarrow \mathbf{w}_I]} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{RB}\left[\mathbf{v}'_I \Rightarrow \mathbf{w}'_I\right]} \\
\tilde{w}: & \textcircled{a} \cup \textcircled{b}_{\mathbf{x}_I, \mathbf{w}_I} & \xrightarrow{\Delta T} & \textcircled{a} \cup \textcircled{b}_{\mathbf{x}'_I \simeq \mathbf{x}_I + \mathbf{w}_I \cdot \Delta T\,,\, \mathbf{w}'_I = \mathbf{w}_I + \Delta \mathbf{w}_I}
\end{array}$$

Both begin in same configuration $\mathbf{x}_I$. $\mathcal{A}$lice stops process w in configuration $\mathbf{x}'_I \simeq \mathbf{x}_I + \mathbf{v}_I \cdot \Delta t$ after duration Δt. She stops other action $\tilde{w}$ in same configuration $\mathbf{x}'_I$, i.e. after running through same configuration changes

$$\mathbf{v}_I \cdot \Delta t \;\stackrel{!}{=}\; \mathbf{w}_I \cdot \Delta T \tag{99}$$

after duration ΔT - corresponding to modified initial conditions $\mathbf{w}_I$.

After steering system $ⓐ \cup ⓑ$ along circular process

$$W \;:=\; \mathrm{RB}\left[\mathbf{v}_I \Rightarrow \mathbf{w}_I\right] \,*\, \tilde{w} \,*\, \mathrm{RB}\left[\mathbf{w}'_I \Rightarrow \mathbf{v}'_I\right] \,*\, w^{-1}$$

the combined calorimeter extract must vanish (85)

$$\begin{aligned} \mathrm{RB}\left[W\right] \;\overset{!}{=}\; 0 \;&=\; \mathrm{RB}\left[\mathbf{v}_I \Rightarrow \mathbf{w}_I\right] \,+\, \mathrm{RB}\left[\mathbf{w}'_I \Rightarrow \mathbf{v}'_I\right] \\ &\overset{(46)}{=}\; \mathrm{RB}\left[\mathbf{v}_I \Rightarrow \mathbf{v}'_I\right] \,-\, \mathrm{RB}\left[\mathbf{w}_I \Rightarrow \mathbf{w}'_I\right] \;. \end{aligned} \tag{100}$$

Hence corresponding energy extract from both infinitesimal segments of intrinsic action w resp. $\tilde{w}$ is the same

$$\begin{aligned} E^{(\mathcal{A})}\left[\mathrm{RB}\left[W\right]\right] \;\overset{!}{=}\; 0 &\overset{(100)(94)}{=} \{m_a\cdot \mathbf{v}_a \cdot \Delta\mathbf{v}_a + m_b\cdot \mathbf{v}_b \cdot \Delta\mathbf{v}_b\} - \{m_a\cdot \mathbf{w}_a \cdot \Delta\mathbf{w}_a + m_b \cdot \mathbf{w}_b\cdot \Delta\mathbf{w}_b\} \\ &= \;\; \{m_a \cdot \Delta\mathbf{v}_a \cdot (\mathbf{v}_a - \mathbf{v}_b)\} \;-\; \{m_a \cdot \Delta\mathbf{w}_a \cdot \underbrace{(\mathbf{w}_a - \mathbf{w}_b)}_{\overset{(99)}{=} \frac{\Delta t}{\Delta T}\cdot(\mathbf{v}_a - \mathbf{v}_b)}\;\} \;. \end{aligned} \tag{101}$$

We have eliminated m_b by momentum conservation $m_a^{(\mathcal{A})} \cdot \Delta\mathbf{v}_a^{(\mathcal{A})} = -m_b^{(\mathcal{A})} \cdot \Delta\mathbf{v}_b^{(\mathcal{A})}$ resp. $m_a^{(\mathcal{A})} \cdot \Delta\mathbf{w}_a^{(\mathcal{A})} = -m_b^{(\mathcal{A})} \cdot \Delta\mathbf{w}_b^{(\mathcal{A})}$ (93), (94).

In both infinitesimal actions w resp. $\tilde{w}$ element ⓐ accelerates in the same way

$$\frac{\Delta\mathbf{v}_a}{\Delta t} \;\overset{(101)}{=}\; \frac{\Delta\mathbf{w}_a}{\Delta T} \;.$$

Therefore in both actions the force against element ⓐ

$$\mathbf{F}_a^{(\mathcal{A})}\left[w\right] \;:=\; \frac{\Delta\mathbf{p}_a^{(\mathcal{A})}}{\Delta t}\left[w\right] \;=\; m_a \cdot \frac{\Delta\mathbf{v}_a}{\Delta t} \;=\; m_a \cdot \frac{\Delta\mathbf{w}_a}{\Delta T} \;=\; \frac{\Delta\mathbf{p}_a^{(\mathcal{A})}}{\Delta T}\left[\tilde{w}\right] \;=:\; \mathbf{F}_a^{(\mathcal{A})}\left[\tilde{w}\right]$$

is the same - and analogous for element ⓑ (see Lemma 10).

Consider an *active* boost of entire system $ⓐ_{\mathbf{v}_a} \cup ⓑ_{\mathbf{v}_b}\big|_{\mathbf{x}_a,\mathbf{x}_b}$ from $\mathcal{A}$lice towards $\mathcal{B}$ob. Let $\mathcal{A}$lice move relative to $\mathcal{B}$ob with constant velocity $\mathbf{v}_{\mathcal{A}} = \left\{-\mathbf{v}_a^{(\mathcal{A})} - \frac{\Delta\mathbf{v}_a^{(\mathcal{A})}}{2}\right\} \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{B})}}$. In $\mathcal{B}$ob's frame intrinsic action $w_{\mathcal{B}}$ (97) evolves in an *intrinsically equivalent* way - with same intrinsic duration $\Delta t^{(\mathcal{B})}$, configuration changes $\Delta\mathbf{x}_I^{(\mathcal{B})}$ and acceleration $\Delta\mathbf{v}_I^{(\mathcal{B})}$. Under the condition of Galilei Kinematics $\mathcal{A}$lice measures for $\mathcal{B}$ob's boosted action $w_{\mathcal{B}}$ [35]

$$w_{\mathcal{B}}: \quad ⓐ_{-\frac{\Delta\mathbf{v}_a}{2}} \cup ⓑ_{\mathbf{v}_b - \mathbf{v}_a - \frac{\Delta\mathbf{v}_a}{2}} \;\;\Rightarrow\;\; ⓐ_{+\frac{\Delta\mathbf{v}_a}{2}} \cup ⓑ_{\left(\mathbf{v}_b - \mathbf{v}_a - \frac{\Delta\mathbf{v}_a}{2}\right) + \Delta\mathbf{v}_b} \tag{102}$$

[35]This is a *passive* transformation of physical quantities of same objects in $w_{\mathcal{B}}$ with regard to different observers $\mathcal{A}$lice and $\mathcal{B}$ob.

same values of acceleration $(\Delta t, \Delta\mathbf{v}_a, \Delta\mathbf{v}_b)$ and hence same physical quantity of momentum changes and forces against element $\textcircled{a}$

$$\mathbf{F}_a^{(\mathcal{A})}[w_{\mathcal{B}}] \overset{\text{(Lem.10)}}{=} \mathbf{F}_a^{(\mathcal{B})}[w_{\mathcal{B}}] \overset{\text{(equiv.)}}{=} \mathbf{F}_a^{(\mathcal{A})}[w_{\mathcal{A}}] \tag{103}$$

as for her intrinsic action $w_{\mathcal{A}}$ (97). Similarly let $\mathcal{A}$lice move relative to $\mathcal{C}$harlie with constant velocity $\mathbf{v}_{\mathcal{A}} = \left\{-\mathbf{w}_a^{(\mathcal{A})} - \frac{\Delta\mathbf{w}_a^{(\mathcal{A})}}{2}\right\} \cdot \mathbf{v}_{\mathbf{1}^{(\mathcal{C})}}$. Again $\mathcal{A}$lice measures for $\mathcal{C}$harlie's boosted action $\tilde{w}_{\mathcal{C}}$

$$\tilde{w}_{\mathcal{C}}: \quad \textcircled{a}_{-\frac{\Delta\mathbf{w}_a}{2}} \cup \textcircled{b}_{\mathbf{w}_b - w_a - \frac{\Delta\mathbf{w}_a}{2}} \quad \Rightarrow \quad \textcircled{a}_{+\frac{\Delta\mathbf{w}_a}{2}} \cup \textcircled{b}_{\left(\mathbf{w}_b - w_a - \frac{\Delta\mathbf{w}_a}{2}\right) + \Delta\mathbf{w}_b} \tag{104}$$

same durations, accelerations and forces against element $\textcircled{a}$

$$\mathbf{F}_a^{(\mathcal{A})}[\tilde{w}_{\mathcal{C}}] \overset{(103)}{=} \mathbf{F}_a^{(\mathcal{A})}[w_{\mathcal{B}}] \tag{105}$$

as for $\mathcal{B}$ob's boosted action $w_{\mathcal{B}}$.

$\mathcal{A}$lice measures two intrinsic actions $w_{\mathcal{B}}$ and $\tilde{w}_{\mathcal{C}}$ in same system $\textcircled{a} \cup \textcircled{b}_{\mathbf{x}_I}$. At same initial configuration $\mathbf{x}_I = (\mathbf{x}_a, \mathbf{x}_b)$ she prepares different initial conditions for actions $w_{\mathcal{B}}$ resp. $\tilde{w}_{\mathcal{C}}$

$$\mathbf{v}_I[w_{\mathcal{B}}] \overset{(102)}{=} \Big(\underbrace{-\frac{\Delta\mathbf{v}_a}{2}}_{\simeq \mathbf{0}} \,,\, \underbrace{\mathbf{v}_b - \mathbf{v}_a - \frac{\Delta\mathbf{v}_a}{2}}_{\simeq \mathbf{v}_b - \mathbf{v}_a} \Big)$$

$$\mathbf{v}_I[\tilde{w}_{\mathcal{C}}] \overset{(104)}{=} \Big(\underbrace{-\frac{\Delta\mathbf{w}_a}{2}}_{\simeq \mathbf{0}} \,,\, \underbrace{\mathbf{w}_b - \mathbf{w}_a - \frac{\Delta\mathbf{w}_a}{2}}_{\simeq \mathbf{w}_b - \mathbf{w}_a} \Big) \,.$$

In both cases initially element $\textcircled{a}_{\mathbf{0}}$ is (practically) at rest while element $\textcircled{b}$ is set up with different initial velocity $\mathbf{v}_b - \mathbf{v}_a$ resp. $\mathbf{w}_b - \mathbf{w}_a$.[36] Therefore the force against same (resting) element $\textcircled{a}_{\mathbf{0}}$ (105) in system $\textcircled{a}_{\mathbf{0}} \cup \textcircled{b}_{\mathbf{v}_b - \mathbf{v}_a}\big|_{\mathbf{x}_I}$ and in system $\textcircled{a}_{\mathbf{0}} \cup \textcircled{b}_{\mathbf{w}_b - \mathbf{w}_a}\big|_{\mathbf{x}_I}$ does not depend on the initial velocity of element $\textcircled{b}$. And in reverse the force against element $\textcircled{b}_{\mathbf{v}_b}$ (in intrinsic action w) in system $\textcircled{a}_{\mathbf{0}} \cup \textcircled{b}_{\mathbf{v}_b}\big|_{\mathbf{x}_I}$ with same resting element $\textcircled{a}_{\mathbf{0}}$ in configuration $\mathbf{x}_I$ does not depend on its velocity $\mathbf{v}_b$ (see Lemma 10). In common words: For *same physical 'source'* $\textcircled{a}_{\mathbf{0}}$ (at rest) the force against different 'test-particles' $\textcircled{b}_{\mathbf{v}_b}\big|_{\mathbf{x}_I}$ is velocity $\mathbf{v}_b$ independent.

□

In Galilei Kinematics the force of intrinsic action w in system $\textcircled{a}_{\mathbf{v}_a} \cup \textcircled{b}_{\mathbf{v}_b}$ is independent from the actual velocity of both elements $\mathbf{v}_a$ and $\mathbf{v}_b$. In Definition 11 we can drop to specify its initial condition $\mathbf{v}_I$. Hence invariant quantitative ratio 'force'

$$\mathbf{F}_a^{(\mathcal{A})} := \frac{\Delta\mathbf{p}_a^{(\mathcal{A})}}{\Delta t_a^{(\mathcal{A})}}\left[w\big|_{\mathbf{x}_I, \mathbf{v}_I}\right] \ / \bmod \mathbf{v}_I$$

is a meaningful *derived* physical quantity. Provided physical conditions for the derivation of force are justified - its significance goes beyond an abbreviating notation for the formalism.

[36]For infinitesimal segments of intrinsic action w we have $\Delta\mathbf{v}_{a/b} \ll \mathbf{v}_{a/b}$ and in so called center of mass frame $\mathbf{v}_a$ and $\mathbf{v}_b$ are oriented antiparallel (92), (59).

5.2 Displacement Work

Quantity of potential energy $V_{\text{pot}}[\mathbf{s}_I \Rightarrow \mathbf{s}'_I]$ is a purely configuration dependent function (83).

Definition 12 *Displacement work $\nabla^{(a)} V_{\text{pot}} \cdot \delta \mathbf{s}_a$ is the steering energy required for generating a partial configuration variation $\mathbf{s}_a, \mathbf{s}_b \Rightarrow \mathbf{s}_a + \delta\mathbf{s}_a, \mathbf{s}_b$ for element ⓐ and fixed position for element ⓑ.*

Proposition 8 *In Galilei Kinematics in conservative system $ⓐ \cup ⓑ\big|_{\mathbf{s}_I}$ the displacement work (of steered action on the entire system) for partial configuration variation $\mathbf{s}_I \Rightarrow \mathbf{s}_I + \delta\mathbf{s}_a$*

$$-\nabla^{(a)} V_{\text{pot}} \cdot \delta\mathbf{s}_a \quad = \quad \mathbf{F}_a \cdot \delta\mathbf{s}_a \tag{106}$$

determines the force (of free evolving intrinsic action w) against element ⓐ.

Proof: We realize partial configuration variation $\mathbf{s}_I \Rightarrow \mathbf{s}_I + \delta\mathbf{s}_a$ between initially and finally resting system $ⓐ_{\mathbf{0}} \cup ⓑ_{\mathbf{0}}\big|_{\mathbf{s}_I} \Rightarrow ⓐ_{\mathbf{0}} \cup ⓑ_{\mathbf{0}}\big|_{\mathbf{s}_I+\delta\mathbf{s}_a}$ by means of *steering actions* (see Lemma 7). We control the dynamical process of intrinsic action w by consecutive association of *exterior* steering actions $\text{RB}^{(i)}$ and segments of undisturbed *intrinsic* actions w_i (see picture 19)

$$W := \text{RB}^{(1)} * w_1 * \ldots * \text{RB}^{(n)} * w_n * \text{RB}^{(n)'} \quad .$$

We couple steering actions - for configuration adjustment $\delta\mathbf{s}_a$ for element ⓐ (and fixing ⓑ) - from exterior calorimeter reservoir suitably (individually against elements ⓐ, ⓑ) *between* free evolving segments of intrinsic action w. In two-partite system $ⓐ \cup ⓑ\big|_{\mathbf{s}_I}$ two different processes - steered action W for configuration variation $\delta\mathbf{s}_a$ (partial only in ⓐ)

$$\begin{array}{rcccc} W & : & ⓐ_{\mathbf{0},\,\mathbf{s}_a} \cup ⓑ_{\mathbf{0},\,\mathbf{s}_b} & \Rightarrow & ⓐ_{\mathbf{0},\,\mathbf{s}_a+\delta\mathbf{s}_a} \cup ⓑ_{\mathbf{0},\,\mathbf{s}_b} \\ & & & & \equiv \\ (-\text{RB}\,[\mathbf{v}_I]) * w * \text{RB}\,[\mathbf{v}'_I] & : & ⓐ_{\mathbf{0},\,\mathbf{s}_a} \cup ⓑ_{\mathbf{0},\,\mathbf{s}_b} & \Rightarrow & ⓐ_{\mathbf{0},\,\mathbf{s}_a+\Delta\mathbf{s}_a} \cup ⓑ_{\mathbf{0},\,\mathbf{s}_b+\Delta\mathbf{s}_b} \end{array} \tag{107}$$

and extraction of kinetic effect from isolated action w (without steering in between) - are equivalent if both generate same intrinsic configuration transition

$$\Delta\mathbf{s}_a - \Delta\mathbf{s}_b \quad = \quad \delta\mathbf{s}_a \quad . \tag{108}$$

Both - steered and free evolving - processes (107) can be coupled into a circular process. Hence the combined steering effort $\sum_{i=1}^{n} \text{RB}^{(i)}$ in steered process W is equivalent (85) to the absorption of the kinetic effect from isolated intrinsic action w. The displacement work/ steering energy - which we successively couple throughout steered process W -

$$\begin{aligned} -\nabla^{(a)} V_{\text{pot}} \cdot \delta\mathbf{s}_a \overset{(75)}{:=} E\left[\sum_{i=1}^{n} \text{RB}^{(i)}\right] &\overset{!}{=} E[\underbrace{(-\text{RB}\,[\mathbf{v}_I]) * w * \text{RB}\,[\mathbf{v}'_I]}_{=\text{RB}\left[ⓐ\cup ⓑ_{\mathbf{0},\mathbf{s}_I} \Rightarrow ⓐ \cup ⓑ_{\mathbf{0},\mathbf{s}'_I}\right]}] \\ &= \underbrace{\Delta E_{\text{kin}\,a}}_{\overset{(96)}{=} \mathbf{F}_a \cdot \Delta\mathbf{s}_a} + \underbrace{\Delta E_{\text{kin}\,b}}_{\overset{(96)}{=} \mathbf{F}_b \cdot \Delta\mathbf{s}_b} \overset{(\text{Lem.}10)}{=} \mathbf{F}_a \cdot \underbrace{(\Delta\mathbf{s}_a - \Delta\mathbf{s}_b)}_{\overset{(108)}{=} \delta\mathbf{s}_a} \end{aligned}$$

equals the extracted kinetic energy from intrinsic action w - which we absorb at the end of undisturbed evolution at one go. Therefore the force of free evolving intrinsic action w against element $\textcircled{a}$

$$\mathbf{F}_a \;=\; -\nabla^{(a)} V_{\text{pot}}$$

is determined by the gradient of potential energy under partial configuration changes $\delta \mathbf{s}_a$ (with fixed position $\mathbf{s}_b$ of element $\textcircled{b}$).

For n-body system $G_1 \cup \ldots \cup G_N$ we disassemble potential energy $V_{G_1 \cup \ldots \cup G_N} := \sum_{i<j} V_{G_i \cup G_j}$ into the potential energy of two-partite systems $V_{G_i \cup G_j}$ - which according to Superposition Principle are independent from the presence of further elements G_k for $k \neq i, j$. Then the force of intrinsic action w against element G_i is given by

$$\mathbf{F}_i \;=\; -\sum_{j \neq i} \nabla^{(i)} V_{G_i \cup G_j} \quad .$$

□

5.3 Equation of Motion

We specify intrinsic action w in conservative system $\textcircled{a}_{\mathbf{v}_a} \cup \textcircled{b}_{\mathbf{v}_b}\big|_{\mathbf{x}_a, \mathbf{x}_b}$ - in complete abstraction from inner dynamics with pre-theoretic ordering relations {2.3}, quantification scheme {3.4.3} and action principles {4} - by basic physical quantities of energy and momentum. Force $\mathbf{F}_i$ specifies for infinitesimal segments of intrinsic action $w : t, \mathbf{s}_I \Rightarrow t + \Delta t, \mathbf{s}_I + \Delta \mathbf{s}_I$ how momentum (95) (and energy (96)) evolves for all elements $i \in I$ in the system.

$$\begin{array}{ccccc} \mathbf{p}_i^{(\mathcal{A})}\big|_{t+\Delta t} & \overset{\text{(Theo.4)}}{=:} & \underbrace{\mathbf{F}_i^{(\mathcal{A})}\Big[w\big|_{\mathbf{v}_I}\Big] \;/\, \mathrm{mod}\ \mathbf{v}_I}_{\overset{\text{(Prop.8)}}{=} \, -\nabla^{(i)} V_{\text{pot}}\big|_{\mathbf{x}_I}} \cdot \Delta t & + & \mathbf{p}_i^{(\mathcal{A})}\big|_t \\ \Big\downarrow \mathrm{RB}_i & & & & \Big\downarrow \mathrm{RB}_i \\ m_i^{(\mathcal{A})} \cdot \left(\mathbf{v}_i^{(\mathcal{A})} + \Delta \mathbf{v}_i^{(\mathcal{A})}\right) & & & & m_i^{(\mathcal{A})} \cdot \mathbf{v}_i^{(\mathcal{A})} \end{array} \tag{109}$$

Since force $\mathbf{F}_i$ of intrinsic action $w\big|_{\mathbf{v}_I}$ has an invariant physical quantity[37] we can *abstract* onto momentum differences despite varying initial conditions $\mathbf{v}_I$. For all momentary velocities the proportionality factor $\mathbf{F}/\mathrm{mod}\ \mathbf{v}_I$ is preserved. We *substitute* the physical quantity of momentum (resp. energy) by kinematical quantities (94) and inherit the equation of motion

$$m_i \cdot \frac{\mathrm{d}^2 \mathbf{s}_i}{\mathrm{d}t^2} \;=\; -\nabla^{(i)} V_{\text{pot}} \qquad \forall\, i \in I \tag{110}$$

for all individual elements. From the differentiated specification of momentum evolution (109) we *induce* Newton's equations for the evolution of motion. Displacement work (106)

[37]Physical quantity of force $\mathbf{F}_i^{(\mathcal{A})}\Big[w\big|_{\mathbf{v}_I}\Big] \,/\, \mathrm{mod}\ \mathbf{v}_I$ is independent of initial condition $\mathbf{v}_I$ (see Theorem 4) and given by gradient of displacement work $-\nabla^{(i)} V_{\text{pot}}$ at momentary configuration $\mathbf{x}_I$ (see Proposition 8).

associated with actual motion of the system $(\Delta t, \Delta \mathbf{s}_I)\big|_t$ - due to inertia and initial velocity $\mathbf{v}_I$ - successively causes infinitesimal variations of the evolving state of motion $(\Delta t, \Delta \mathbf{s}_I)\big|_{t+\Delta t}$ throughout the course of intrinsic action w.

Next consider mechanical systems with - built in - constraints. We analyze spatiotemporal evolution of intrinsic action w in closed system $E \cup G_1 \cup \ldots \cup G_N$. In addition we subdivide into external element E (e.g. earth) and 'inner' elements $G_1 \cup \ldots \cup G_N$ (e.g. bound parts of a physical pendulum). The latter have fixed rigid connections among one another. They enforce a partial fixation of sought after solution to the equations of motion $\mathbf{s}_i - \mathbf{s}_j \overset{!}{=}$ const. for all inner elements $i, j \in I := \{1, \ldots, N\}$. The motion of rigid subsystem $G_1 \cup \ldots \cup G_N$ must be compatible with n (holonomic) constraints. We parameterize admissible displacements (degrees of freedom of the subsystem) by $3N - n$ generalized coordinates q

$$\Delta \mathbf{s}_I \;=\; \sum_{k=1}^{3N-n} \frac{\partial \mathbf{s}_I}{\partial q_k} \cdot \Delta q_k \quad .$$

D'Alembert postulates: (*inner*) constraint forces do not contribute to displacement work. For (admissible) inertial displacements - due to given initial velocity $\dot{q}\big|_t$ - the evolution of energy (96)

$$\Delta E_{\text{kin } I} \;=\; -\nabla^{(q)} V_{\text{pot}} \cdot \Delta q \qquad (111)$$

is determined by (*applied* forces from) the potential $V_{\text{pot}} = \sum_{i \in I} V_{E \cup G_i}$ of inner parts of the subsystem G_i with external element E. Substitution of physical quantity of kinetic energy by kinematical quantities (94) induces (Lagrange's form of) the equations of motion

$$\begin{aligned} 0 \;&\overset{(111)}{=}\; \sum_{i=1}^{N} m_i \cdot \underbrace{\mathbf{v}_i}_{\approx \frac{\Delta \mathbf{s}_i}{\Delta t}} \cdot \underbrace{\Delta \mathbf{v}_i}_{= \mathbf{a}_i \cdot \Delta t} \;+\; \underbrace{\nabla^{(q_K)} V_{\text{pot}}}_{= \nabla^{(\mathbf{s}_I)} V_{\text{pot}} \cdot \frac{\partial \mathbf{s}_I}{\partial q_K}} \cdot \Delta q_K \\ &=\; \sum_{i=1}^{N} \big(m_i \cdot \mathbf{a}_i \;+\; \nabla^{(i)} V_{\text{pot}} \big) \cdot \underbrace{\frac{\partial \mathbf{s}_i}{\partial q_K} \cdot \Delta q_K}_{=: \delta \mathbf{s}_i} \end{aligned}$$

for all admissible - so called - 'virtual displacements' $\delta \mathbf{s}_I$. D'Alembert's 'Principle of virtual work' is an additional postulate to account for the collective effect of constraint forces. Without determining the details of unknown inner binding actions this method provides *reduced equations of motion* for rigid bound subsystem - in generalized coordinates.[38]

[38] Ruben outlines the historic development [14]: Newton's Mechanics is essentially formulated for the free point mass. The formation of mechanics for systems was stimulated by the question of Mersenne 1646, to determine the center of oscillation for a physical pendulum. The difficulty was that (constraint) forces - due to rigid spatial connections - were unknown. They had to be determined from their 'effect'. Huygens 1673 recognized the law according to which different elements of the composite pendulum mutually influence their motion which is driven by respective force of gravity: If several weights which are attached to a pendulum fall

6 Principle of Least Action

6.1 Historic Development

Szabó [27] outlines the historic development. Fermat 1629 has been first to take a general principle as a basis for motion: "the only requirement is that nature always proceeds on the way of least resistance... but not, as people generally assume, that nature always chooses the shortest way". Leibniz 1708 introduced a quantity of action and explained: "The action is not what you think, here the consideration of time is inevitable; the action is like the product of $\{m, \mathbf{s}, \mathbf{v}\}$ or of $\{t, E_{\text{kin}}\}$. I have noticed that during changes of motion it always turns into a maximum or minimum." Euler 1743 gave this proposition a mathematically immaculate form. He recognized that: "all actions in nature obey some law of maximum or minimum... Some attribute of maximum or minimum is localized in the trajectory of thrown objects... The nature of this property can not be seen easily from metaphysical principles. The trajectories in question are ascertainable by direct methods (with less calculus)... so that one can determine what of them is maximal or minimal. (Euler particularly regards) the resulting effect of acting forces on the state of motion of bodies... I did not discover these interesting connections a priori but only a posteriori... after several attempts... I found the expression for a quantity which turns into a minimum during natural motion" [3].

While Leibniz (just as Maupertuis) suggests a teleological guiding principle: "that the actual world is the best of all possible worlds" - Euler ascribes to his quantity of action no further validity beyond the examined cases. According to Euler no general principle is found. Bavink [26] describes the teleological interpretation: "It seems as if nature selects from *many per se possible* motions the one which achieves a largest possible effect through least possible means... To the present day Hamilton's Principle has to serve for lines of thought... which see processes evolve, as if so to say *nature had to consider* at the beginning of time period $t_2 - t_1$ how to keep the value of an *integral* $\int_{t_1}^{t_2} E\mathrm{d}t$ as low as possible." Bavink diagnoses "these propositions essentially contain nothing but the statement, that under certain circumstances something certain happens: the principle of causality. What really happens is determined by differential equations and every differential equation can be regarded as a condition that a certain function turns into a minimum or maximum - in the latter one even has a wide freedom of choice. With the validity of equations of motion for Mechanics one can theoretically state many other functions of this sort."

What in Bavink's purely mathematical point of view appears - as a sober statement of facts, as complete equivalence of differential principle and integral principle (about *physical quantities*) - we reconsider taking into account also their *physical conditions*. In reality what happens is determined by equations of motion {5.3}. Given initial conditions $\mathbf{s}_I, \mathbf{v}_I$ permit - not many but - exactly one course of motion. To realize many other possible processes

then their center of gravity swings back to same height independently whether the rigid spatial connections are separated or not. Huygens solution is based on the Principle of impossibility of a perpetuum mobile. D'Alembert 1743 introduced the differential formulation of this principle and Lagrange - using the concept of work - brought it into the familiar form: Constraint forces do not provide work! [7]

requires external steering interventions. In order to run along different paths γ resp. $\gamma + \delta\gamma$ through same end point configurations $\mathbf{s}_I \Rightarrow \mathbf{s}'_I$ in fixed duration Δt - not nature but - steering physicist needs to consider how to couple temporary steering actions $\mathrm{RB}^{(i)}$. He has the choice between many possible (types of) steering options. The integral named 'coercion' or 'action' has physical meaning only in its role as variation functional. A variation essentially analyzes the difference between two - differently steered - processes of an action.

Planck [11] emphasizes to define a variational principle requires stipulation of

- *conditions* for 'virtual motions', i.e. the repertory of processes *from which to choose* (resp. what steerable processes of interactions of motion we compare with one another)
- quantity of 'action', i.e. the *characteristic with regard to which the selection is made.*

"The former is of exactly same importance as the quantity of action itself - Planck explains - because depending on the type of stipulated variation conditions the content of the principle takes a completely different meaning! [39] It took a long time until that - long disregarded - circumstance was understood clearly and let after precursors Leibniz, Maupertuis, Euler to the first correct version of the Principle of Least Action."

Lagrange 1760 compares motions in a system of material points γ resp. $\gamma + \delta\gamma^{(\mathrm{Lagr})}$ between fixed endpoint configurations $\mathbf{s}_I$ and $\mathbf{s}'_I$ under the condition that the quantity of total energy $E_{\mathrm{tot}} \overset{!}{=} \mathrm{const}$ does not change. On the contrary he permits arbitrary variations in duration $t\big[\gamma + \delta\gamma^{(\mathrm{Lagr})}\big]$. Then - Lagrange postulates - his quantity of action

$$S_{\mathrm{Lagr}}\,[\gamma] \;:=\; \int_{\gamma\big|_{E_{\mathrm{tot}}}} E_{\mathrm{kin}\,I}\,\mathrm{d}t \;=\; \int_{\gamma\big|_{E_{\mathrm{tot}}}} \frac{1}{2}\,m_I\,\mathbf{v}_I\cdot\mathbf{v}_I\,\mathrm{d}t \;=\; \frac{1}{2}\int_{\gamma\big|_{E_{\mathrm{tot}}}} \mathbf{p}_I\cdot\mathrm{d}\mathbf{s}_I \qquad (112)$$

becomes minimal for the true trajectory (i.e. free running process γ of intrinsic action w). An engineer can generate Lagrange variations $\delta\gamma^{(\mathrm{Lagr})}$ of the course of intrinsic action w by suitable steering actions: e.g. instantaneous redistribution (absorbing here, expending there) of energetic units $\mathbf{1}_E\big|_{\mathbf{0}}$ between different elements of the system. He can temporarily couple momentum units $\mathbf{1}_{\mathbf{p}}$ from his external reservoir into elastic collisions against the system but no extra units of energy.

Similarly Hamilton 1834 compares trajectories γ resp. $\gamma + \delta\gamma^{(\mathrm{Ham})}$ between fixed endpoint configurations $\mathbf{s}_I$ and $\mathbf{s}'_I$ but he requires that duration $\Delta t \overset{!}{=} \mathrm{const}$ is preserved. Instead he allows temporary variations of total energy $E_{\mathrm{tot}}\big[\gamma + \delta\gamma^{(\mathrm{Ham})}\big]$. Then - Hamilton postulates - his quantity of action

$$S_{\mathrm{Ham}}\,[\gamma] \;:=\; \int_{\gamma\big|_{\Delta t}} \left(E_{\mathrm{kin}\,I} - V_{\mathrm{pot}}\right)\mathrm{d}t \qquad (113)$$

becomes minimal for the true (undisturbed, isolated) motion of the system.[40] In order to generate - Hamilton's type of - variations $\delta\gamma^{(\mathrm{Ham})}$ requires other steering actions. The engi-

[39] We cannot make the quantity of 'action' into an absolute independently from its selection conditions.

[40] Helmholtz called Hamilton's integrand (113) 'kinetic potential' - nowadays it is called 'Lagrangian'.

neer can temporarily expend additional units of energy $\mathbf{1}_E\big|_{\mathbf{0}}$ (from his external calorimeter reservoir) but he needs to keep an eye on steering duration.[41]

6.2 Minimal Steering Effort

We *analyze* the *variation of* the course of intrinsic *action* w from a perspective which includes familiar mathematical formulation but also those aspects which have long been neglected. We reconsider its methodical and physical conditions. So called 'virtual displacements' $\gamma+\delta\gamma$ become real physical processes. The engineer varies the course of intrinsic action w by coupling extrinsic steering actions $\mathrm{RB}^{(1)} * w_1 * \mathrm{RB}^{(2)} * w_2 * \ldots$ (see Remark 17). We compare two real physical processes - free actions and steered actions - with one another with regard to the steering effort required for generating (temporary) deviation $\delta\gamma$.

For illustration we revisite {4.1} contraction action w of a charged spring with massive bodies $ⓐ \cup ⓑ$ attached on both ends (see figure 22a). Without external steering interventions intrinsic action w evolves from initially expanded configuration $(x_1, t_1) \Rightarrow (x_2, t_2) \Rightarrow (x_3, t_3)$ with increasing relative velocity towards more contracted final configuration (x_3, t_3). An engineer can also steer the course of contraction action w through a varied intermediate state $(x_1, t_1) \Rightarrow (x_2 + \delta x_2, t_2) \Rightarrow (x_3, t_3)$. At same initially resting configuration (x_1, t_1) a practically instantaneous kick $\mathrm{RB}^{(1)} : ⓐ_{\mathbf{0}}, ⓑ_{\mathbf{0}} \Rightarrow ⓐ_{\tilde{\mathbf{v}}_1}, ⓑ_{-\tilde{\mathbf{v}}_1}$ catapults both objects into opposite motion. Under modified initial conditions contraction action $\tilde{w}_1$ evolves towards more expanded intermediate state $(x_2 + \delta x_2, t_2)$. Another steering kick of the right strength $\mathrm{RB}^{(2)} : ⓐ_{\tilde{\mathbf{v}}_{2-}}, ⓑ_{-\tilde{\mathbf{v}}_{2-}} \Rightarrow ⓐ_{\tilde{\mathbf{v}}_{2+}}, ⓑ_{-\tilde{\mathbf{v}}_{2+}}$ provides enough momentum so that consecutive segment of contraction action $\tilde{w}_2$ evolves to same final configuration (x_3, t_3) - with more velocity though. In final steering action $\mathrm{RB}^{(3)}$ he extracts surplus kinetic energy and momentum from both elements ⓐ, ⓑ such that - after completion $\mathrm{RB}^{(1)} * \tilde{w}_1 * \mathrm{RB}^{(2)} * \tilde{w}_2 * \mathrm{RB}^{(3)}$ - the system continues evolving like contraction action w in the same undisturbed way.

By coupling three external steering resp. absorption kicks $\mathrm{RB}^{(1)}, \mathrm{RB}^{(2)}, \mathrm{RB}^{(3)}$ at the right moment and at the right strength the engineer generates - Hamilton's type of - variation $\gamma + \delta\gamma^{(\mathrm{Ham})}$ of the course of intrinsic action w (see figure 22b). The system of both massive bodies and spring $ⓐ \cup ⓑ$ runs through fixed endpoint configurations $\mathbf{x}_1 := (\mathbf{x}_a, \mathbf{x}_b)\big|_{t_1}$ and $\mathbf{x}_3 := (\mathbf{x}_a, \mathbf{x}_b)\big|_{t_3}$ in same fixed duration $\Delta t := t_3 - t_1$. To steer the course of action through varied intermediate configuration $\mathbf{x}_2 + \delta\mathbf{x}_2 := (\mathbf{x}_a, \mathbf{x}_b)\big|_{t_2} + (\delta\mathbf{x}_a, \delta\mathbf{x}_b)$ an engineer temporarily expends steering energy $E_{\mathrm{RB}^{(1)}}\big|_{t_1}$ and $E_{\mathrm{RB}^{(2)}}\big|_{t_2}$ out of his external calorimeter reservoir - which he fully retrieves $E_{\mathrm{RB}^{(3)}}\big|_{t_3}$ at the end of his steering maneuver according to (85).

That steered contraction action of charged spring $\mathrm{RB}^{(1)} * \tilde{w}_1 * \mathrm{RB}^{(2)} * \tilde{w}_2 * \mathrm{RB}^{(3)}$ is a simple representative for a physical process behind Hamilton's variation of action w (see figure 23). Both processes run through fixed endpoints $\mathbf{x}_1$ and $\mathbf{x}_3$ in fixed duration Δt.

[41] In the special case of no driving forces the free mass point runs through fixed endpoint configuration $\mathbf{s}_I \Rightarrow \mathbf{s}'_I$ - according to Lagrange (112) with constant velocity $|\mathbf{v}_I|$ in minimum duration t and according to Hamilton (113) with minimum velocity $|\mathbf{v}_I|$ in fixed duration Δt - along the shortest path, a *straight line.*

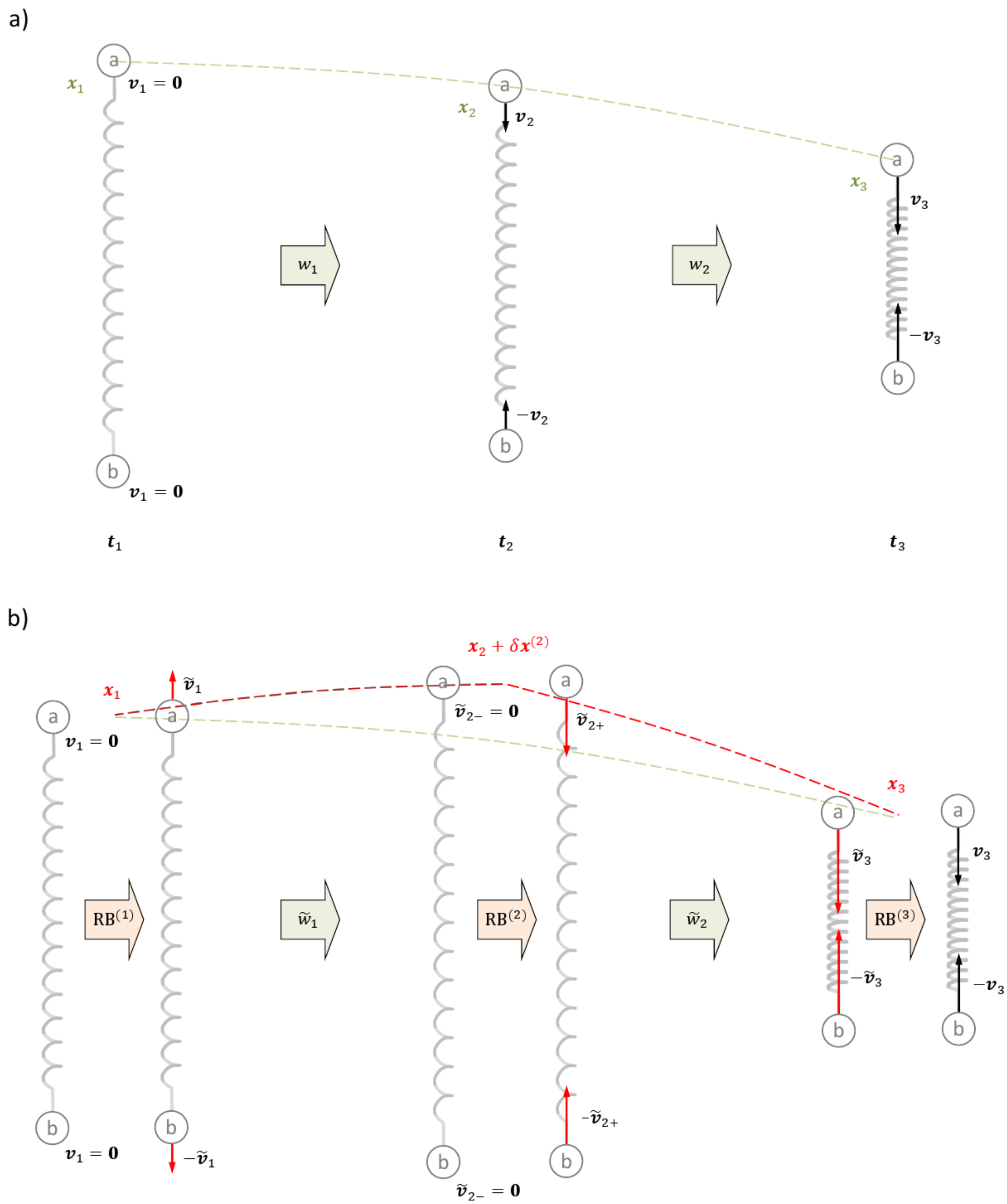

Figure 22: a) contraction action b) course steered by instantaneous calorimeter interventions

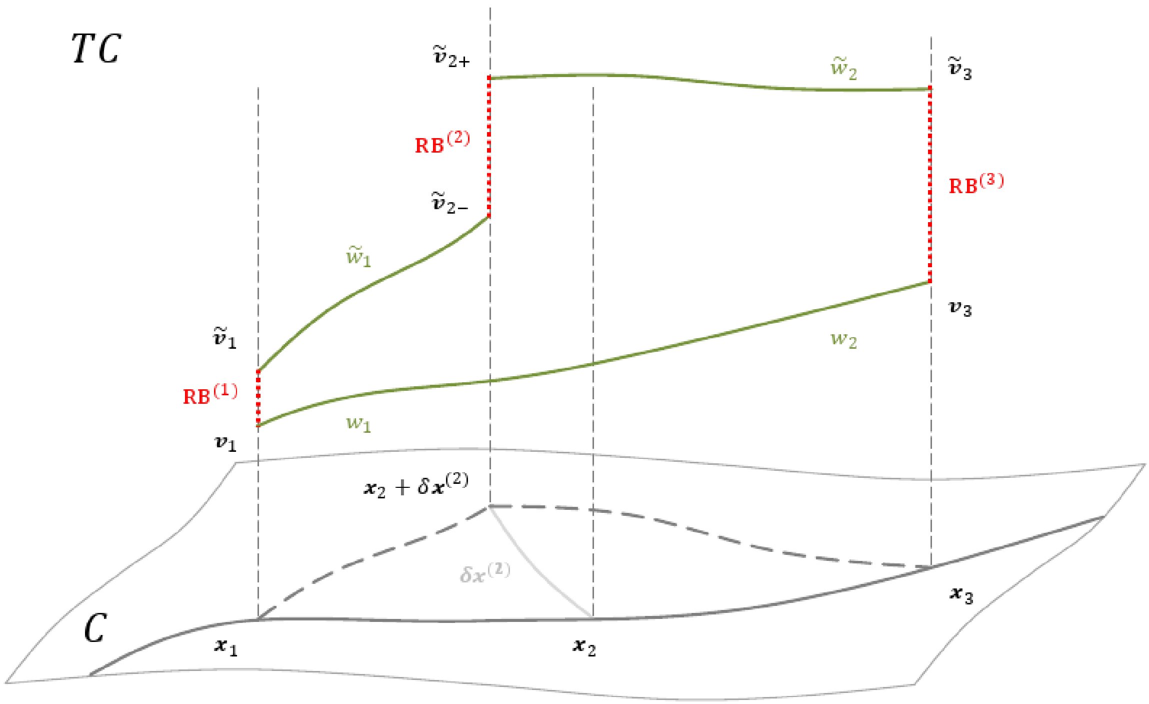

Figure 23: controlled variation of action w

Despite varied intermediate configuration $\mathbf{x}_2 + \delta\mathbf{x}_2$ - after completing temporary steering intervention - both processes w resp. $\mathrm{RB}^{(1)} * \tilde{w}_1 * \mathrm{RB}^{(2)} * \tilde{w}_2 * \mathrm{RB}^{(3)}$ continue evolving in the same undisturbed way. An engineer can successively substitute free segments of intrinsic action $\tilde{w}_i$ with 'three-step' steering actions and thus generate every form for a local Hamilton type variation $\delta\gamma^{(\mathrm{Ham})}$. Bernoulli summarized the basic idea of variation: "The extremal property of sought after curve is also contained in all segments of that curve, in particular also in all of its (infinitesimal) elements" [27]. Hence for our variational analysis (of the steering effort) it is sufficient to examine the physical process behind basic 'three-step' variation maneuver of figure 23.

Theorem 5 *For every local Hamilton type variation $\delta\gamma^{(\mathrm{Ham})}$ - steered through fixed endpoint configurations $\mathbf{x}_I$ and $\mathbf{x}'_I$ in fixed duration Δt - of the free evolution γ of intrinsic action w the variation of Hamilton's quantity of action (113) is positive definite*

$$0 \;<\; \delta S_{\mathrm{Ham}}\left[\gamma\right] \;:=\; S_{\mathrm{Ham}}\left[\gamma + \delta\gamma^{(\mathrm{Ham})}\right] \;-\; S_{\mathrm{Ham}}\left[\gamma\right] \qquad \forall\;\; \delta\gamma^{(\mathrm{Ham})}. \tag{114}$$

Proof: In the spirit of Euler {6.1} we examine the extremal characteristic - the quantity which turns into a minimum for undisturbed actions - from the direct method. In a system of point masses $\{G_I\}$ the trajectory of intrinsic action w is determined by equations of motion

$$m_I \cdot \frac{\mathrm{d}^2\mathbf{x}_I}{\mathrm{d}t^2} \;\overset{(110)}{=}\; -\nabla^{(I)} V_{\mathrm{pot}} \;\; .$$

For variations of Bernoulli's infinitesimal nature the force against respective elements $i \in I$

$$-\nabla^{(i)} V_{\text{pot}}\Big|_{\mathcal{U}_{\mathbf{x}_I}} \;\simeq\; -\nabla^{(i)} V_{\text{pot}}\Big|_{\mathbf{x}_I}$$

is constant in local neighborhood $\mathcal{U}$ of initial configuration $\mathbf{x}_I$. Without restricting generality we analyze 'three-step' steering maneuver for generating Hamilton type variations - of the course of intrinsic action w - with particular focus on temporarily expended steering actions $\text{RB}^{(i)}$ and corresponding steering duration ΔT_i.[42]

For given initial configuration $(T_1, \mathbf{x}_I)$ and initial velocity $\left\{\frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}t}\right\}_{i\in I} =: \frac{\mathrm{d}\mathbf{x}_I}{\mathrm{d}t} =: \mathbf{v}_I$ resp. $\tilde{\mathbf{v}}_I$ the *free course* of intrinsic action w has local trajectory for elements $i \in I$ of the system

$$(t, \mathbf{x}_I) \;=\; (T_1, \mathbf{x}_I) \;+\; (1, \mathbf{v}_I)\cdot(t - T_1) \;-\; \left(0, \frac{\nabla^I V_{\text{pot}}}{m_I}\right)\cdot\frac{1}{2}\cdot(t-T_1)^2 \;. \tag{115}$$

Let steering start at moment $T_1 = 0$. We define *'three-step' variation* of intrinsic action w - through fixed endpoints $\mathbf{x}_I\big|_{T_1}$, $\mathbf{x}_I\big|_{T_3}$ in fixed duration $(T_3 - T_1)$ - in terms of given deviation $\mathbf{x}_I\big|_{T_2} + \delta\mathbf{x}_I^{(2)}$ from the free trajectory at intermediate moment T_2 (see figure 23).

An engineer plans required steering actions - in direct method - by successive substitution

1. Given configuration of 'three-step' variation $\delta\gamma^{(\text{Ham})}(T_1, T_2, T_3, \delta\mathbf{x}_I^{(2)})$ implies matching conditions between segments of undisturbed intrinsic action w, $\tilde{w}_1$ and $\tilde{w}_2$.

2. corresponding trajectories of w, $\tilde{w}_1$ and $\tilde{w}_2$ specify the transition between respective initial velocity $\mathbf{v}_I^{(1)} \Rightarrow \tilde{\mathbf{v}}_I^{(1)}$ at steering moment T_1 and analogous for T_2, T_3

3. transition $\mathbf{v}_I^{(1)} \Rightarrow \tilde{\mathbf{v}}_I^{(1)}$ determines the strength of required steering kick $\text{RB}^{(1)}$ etc.

We determine steering action $\text{RB}^{(1)} : \mathbf{v}_I^{(1)} \Rightarrow \tilde{\mathbf{v}}_I^{(1)}$ to prepare initial velocity for action $\tilde{w}_1$ by displacement condition $\tilde{\mathbf{x}}_I\big|_{T_2} \stackrel{!}{=} \mathbf{x}_I\big|_{T_2} + \delta\mathbf{x}_I^{(2)}$. Corresponding local free trajectories

$$\begin{aligned}
\mathbf{x}_I + \tilde{\mathbf{v}}_I^{(1)}\cdot T_2 - \frac{\nabla^I V_{\text{pot}}}{m_I}\cdot\frac{1}{2}\cdot {T_2}^2 \;&\stackrel{(115)}{=}\; \mathbf{x}_I + \mathbf{v}_I^{(1)}\cdot T_2 - \frac{\nabla^I V_{\text{pot}}}{m_I}\cdot\frac{1}{2}\cdot {T_2}^2 \;+\; \delta\mathbf{x}_I^{(2)} \\
\tilde{\mathbf{v}}_I^{(1)} \;&\stackrel{!}{=}\; \mathbf{v}_I^{(1)} + \frac{\delta\mathbf{x}_I^{(2)}}{T_2}
\end{aligned} \tag{116}$$

specify the required transition of initial velocities. At starting moment T_1 the engineer expends '$-$'/absorbs '+' steering energy from/into his *external calorimeter reservoir*

$$E_{\text{RB}^{(1)}} \;\stackrel{(59)}{=}\; \frac{1}{2}\cdot m_I\cdot\left\{\left(\mathbf{v}_I^{(1)}\right)^2 - \left(\tilde{\mathbf{v}}_I^{(1)}\right)^2\right\} \tag{117}$$

[42] We analyze intrinsic action w in closed system $\{G_I\}$. Potential energy and force do not depend on time.

where we *sum* all contributions for individual elements $i \in I$ of the system.[43]

Similarly we find final steering action $\mathrm{RB}^{(3)} : \tilde{\mathbf{v}}_I^{(3)} \Rightarrow \mathbf{v}_I^{(3)}$ for absorbing surplus kinetic energy and momentum from $\tilde{w}_2$. We reexpress local trajectories (115) for w and $\tilde{w}_2$ with reference to *joint* configuration $(T_3, \mathbf{x}_I)$ and final velocity $\mathbf{v}_I^{(3)}$ resp. $\tilde{\mathbf{v}}_I^{(3)}$

$$(t, \mathbf{x}_I) \;=\; (T_3, \mathbf{x}_I) \;+\; \left(1, \mathbf{v}_I^{(3)}\right) \cdot (t - T_3) \;-\; \left(0, \frac{\nabla^I V_{\mathrm{pot}}}{m_I}\right) \cdot \frac{1}{2} \cdot (t - T_3)^2 \;\; . \tag{118}$$

Again displacement condition $\tilde{\mathbf{x}}_I\big|_{T_2} \overset{!}{=} \mathbf{x}_I\big|_{T_2} + \delta\mathbf{x}_I^{(2)}$ on corresponding undisturbed trajectories

$$\begin{aligned} \mathbf{x}_I^{(3)} + \tilde{\mathbf{v}}_I^{(3)} \cdot (T_2 - T_3) - \frac{\nabla^I V_{\mathrm{pot}}}{m_I} \cdot \frac{1}{2} \cdot (T_2 - T_3)^2 & \\ = \;\; \mathbf{x}_I^{(3)} + \mathbf{v}_I^{(3)} \cdot (T_2 - T_3) - \frac{\nabla^I V_{\mathrm{pot}}}{m_I} \cdot \frac{1}{2} \cdot (T_2 - T_3)^2 \;+\; \delta\mathbf{x}_I^{(2)} & \\ \tilde{\mathbf{v}}_I^{(3)} \;\; \overset{!}{=} \;\; \mathbf{v}_I^{(3)} + \frac{\delta\mathbf{x}_I^{(2)}}{T_2 - T_3} & \end{aligned} \tag{119}$$

specifies the required transition between final velocities for every element $i \in I$ of the system. The engineer retrieves all his - temporarily expended - steering energy at final moment T_3

$$E_{\mathrm{RB}^{(3)}} \;=\; \frac{1}{2} \cdot m_I \cdot \left\{ \left(\tilde{\mathbf{v}}_I^{(3)}\right)^2 - \left(\mathbf{v}_I^{(3)}\right)^2 \right\} \;\; . \tag{120}$$

Finally we determine required steering kick $\mathrm{RB}^{(2)} : \tilde{\mathbf{v}}_I^{(2-)} \Rightarrow \tilde{\mathbf{v}}_I^{(2+)}$ for middle turning from free evolving segment $\tilde{w}_1$ to $\tilde{w}_2$ (see figure 23). Derivation of left resp. right coming trajectories at turning moment T_2 gives corresponding velocities

$$\begin{aligned} \tilde{\mathbf{v}}_I^{(2-)} & \overset{(116)}{=} \mathbf{v}_I^{(2)} + \frac{\delta\mathbf{x}_I^{(2)}}{T_2} \\ \tilde{\mathbf{v}}_I^{(2+)} & \overset{(119)}{=} \mathbf{v}_I^{(2)} + \frac{\delta\mathbf{x}_I^{(2)}}{T_2 - T_3} \;\; . \end{aligned}$$

Here the engineer expends '$-$'/extracts '+' additional units of energy into/from *system* $\{G_I\}$

$$E_{\mathrm{RB}^{(2)}} \;=\; \frac{1}{2} \cdot m_I \cdot \left\{ \left(\tilde{\mathbf{v}}_I^{(2-)}\right)^2 - \left(\tilde{\mathbf{v}}_I^{(2+)}\right)^2 \right\} \;\; . \tag{121}$$

Given trajectories for undisturbed segments of intrinsic action w, $\tilde{w}_1$ and $\tilde{w}_2$ we examine Hamilton's quantity of action

$$S_{\mathrm{Ham}}\left[\gamma\right] \overset{(113)}{=} \int_{\gamma|_{\Delta t}} \mathrm{d}t \; \left\{ \frac{m_I}{2} \cdot \left(\mathbf{v}_I\big|_t\right)^2 \;-\; V_{\mathrm{pot}}\big|_{\mathbf{x}_I(t)} \right\}$$

[43] We do not need to keep track of associated steering momentum $\mathbf{p}_{\mathrm{RB}^{(1)}} := \mathbf{p}\left[\mathrm{RB}[\mathbf{v}_I^{(1)} \Rightarrow \tilde{\mathbf{v}}_I^{(1)}]\right]$ here.

with summation convention for kinetic energy of all elements $i \in I$. *Locally* around initial configuration $\mathbf{x}_I^{(1)} := \mathbf{x}_I\big|_{T_1}$ the potential field $V_{\text{pot}}\big|_{\mathbf{x}_I} \simeq V_{\text{pot}}\big|_{\mathbf{x}_I^{(1)}} + \nabla^I V_{\text{pot}} \cdot [\mathbf{x}_I - \mathbf{x}_I^{(1)}]$ is linear. For *undisturbed course* γ of intrinsic action w we obtain

$$S_{\text{Ham}}[\gamma] \;=\; \int_{T_1}^{T_3} \mathrm{d}t \;\left\{\frac{m_I}{2}\cdot\left(\mathbf{v}_I\big|_t\right)^2 \;-\; \left(V_{\text{pot}}^{(1)} + \nabla^I V_{\text{pot}}\cdot[\mathbf{x}_I\big|_t - \mathbf{x}_I^{(1)}]\right)\right\}$$

and similarly for *steered course* $\tilde{\gamma}_1 * \tilde{\gamma}_2$ of 'three-step' variation $\text{RB}^{(1)} * \tilde{w}_1 * \text{RB}^{(2)} * \tilde{w}_2 * \text{RB}^{(3)}$

$$\begin{aligned} S[\tilde{\gamma}_1 * \tilde{\gamma}_2] \;=\; & \int_{T_1}^{T_2} \mathrm{d}t \left\{\frac{m_I}{2}\cdot\left(\mathbf{v}_I\big|_t + \frac{\delta\mathbf{x}_I^{(2)}}{T_2}\right)^2 - \left(V_{\text{pot}}^{(1)} + \nabla^I V_{\text{pot}}\cdot[\mathbf{x}_I\big|_t + \frac{\delta\mathbf{x}_I^{(2)}}{T_2}\cdot t - \mathbf{x}_I^{(1)}]\right)\right\} \\ & + \int_{T_2}^{T_3} \mathrm{d}t \left\{\frac{m_I}{2}\cdot\left(\mathbf{v}_I\big|_t + \frac{\delta\mathbf{x}_I^{(2)}}{T_2 - T_3}\right)^2 - \left(V_{\text{pot}}^{(1)} + \nabla^I V_{\text{pot}}\cdot[\mathbf{x}_I\big|_t + \frac{\delta\mathbf{x}_I^{(2)}}{T_2 - T_3}\cdot(t - T_3) - \mathbf{x}_I^{(1)}]\right)\right\} \end{aligned}$$

with varied trajectories $\tilde{\mathbf{x}}_I$, $\tilde{\mathbf{v}}_I$ substituted in terms of

$$\tilde{\mathbf{x}}_I\big|_{[T_1,T_2]} \overset{(115)(116)}{=} \mathbf{x}_I\big|_t \;+\; \frac{\delta\mathbf{x}_I^{(2)}}{T_2}\cdot t \qquad \text{resp.} \qquad \tilde{\mathbf{x}}_I\big|_{[T_2,T_3]} \overset{(118)(119)}{=} \mathbf{x}_I\big|_t \;+\; \frac{\delta\mathbf{x}_I^{(2)}}{T_2 - T_3}\cdot(t - T_3) \;\;.$$

The variation of Hamilton's quantity of action

$$\begin{aligned} \delta S_{\text{Ham}}[\gamma] \;:=\; & S_{\text{Ham}}[\tilde{\gamma}_1 * \tilde{\gamma}_2] \;-\; S_{\text{Ham}}[\gamma] \\ =\; & \int_{T_1}^{T_2} \mathrm{d}t \left\{\frac{m_I}{2}\cdot\left(\left(\frac{\delta\mathbf{x}_I^{(2)}}{T_2}\right)^2 + 2\cdot\mathbf{v}_I\big|_t\cdot\frac{\delta\mathbf{x}_I^{(2)}}{T_2}\right) - \;\nabla^I V_{\text{pot}}\cdot\frac{\delta\mathbf{x}_I^{(2)}}{T_2}\cdot t\right\} \\ & + \int_{T_2}^{T_3} \mathrm{d}t \left\{\frac{m_I}{2}\cdot\left(\left(\frac{\delta\mathbf{x}_I^{(2)}}{T_2 - T_3}\right)^2 + 2\cdot\mathbf{v}_I\big|_t\cdot\frac{\delta\mathbf{x}_I^{(2)}}{T_2 - T_3}\right) - \;\nabla^I V_{\text{pot}}\cdot\frac{\delta\mathbf{x}_I^{(2)}}{T_2 - T_3}\cdot(t - T_3)\right\} \\ =:\; & \underbrace{\frac{m_I}{2}\cdot\left(\frac{\delta\mathbf{x}_I^{(2)}}{T_2}\right)^2\cdot(T_2 - T_1)}_{>0} \;+\; \underbrace{\frac{m_I}{2}\cdot\left(\frac{\delta\mathbf{x}_I^{(2)}}{T_2 - T_3}\right)^2\cdot(T_3 - T_2)}_{>0} \;+\; R \end{aligned} \tag{122}$$

decomposes into two positive terms and a residue R which - using $\mathbf{v}_I\big|_t \overset{(115)}{=} \mathbf{v}_I^{(1)} - \frac{\nabla^I V_{\text{pot}}}{m_I}\cdot t$ -

$$\begin{aligned} R \;\overset{(122)}{:=}\; & \int_{T_1=0}^{T_2} \mathrm{d}t \left\{m_I\cdot\mathbf{v}_I^{(1)}\cdot\frac{\delta\mathbf{x}_I^{(2)}}{T_2} \;-\; m_I\cdot\frac{\delta\mathbf{x}_I^{(2)}}{T_2}\cdot\frac{\nabla^I V_{\text{pot}}}{m_I}\cdot t \;-\; \nabla^I V_{\text{pot}}\cdot\frac{\delta\mathbf{x}_I^{(2)}}{T_2}\cdot t\right\} \\ & + \int_{T_2}^{T_3} \mathrm{d}t \left\{m_I\cdot\mathbf{v}_I^{(1)}\cdot\frac{\delta\mathbf{x}_I^{(2)}}{T_2 - T_3} \;-\; \frac{\delta\mathbf{x}_I^{(2)}}{T_2 - T_3}\cdot\nabla^I V_{\text{pot}}\cdot t \;-\; \nabla^I V_{\text{pot}}\cdot\frac{\delta\mathbf{x}_I^{(2)}}{T_2 - T_3}\cdot(t - T_3)\right\} \end{aligned}$$

$$
\begin{aligned}
R \;=\; & \left\{ m_I \cdot \mathbf{v}_I^{(1)} \cdot \frac{\delta \mathbf{x}_I^{(2)}}{T_2} \cdot T_2 \;-\; \frac{\delta \mathbf{x}_I^{(2)}}{T_2} \cdot \nabla^I V_{\text{pot}} \cdot T_2^2 \right\} \;+\; \left\{ m_I \cdot \mathbf{v}_I^{(1)} \cdot \frac{\delta \mathbf{x}_I^{(2)}}{T_2 - T_3} \cdot (T_3 - T_2) \right. \\
& \left. \;-\; \frac{\delta \mathbf{x}_I^{(2)}}{T_2 - T_3} \cdot \nabla^I V_{\text{pot}} \cdot (T_3^2 - T_2^2) \;-\; \nabla^I V_{\text{pot}} \cdot \frac{\delta \mathbf{x}_I^{(2)}}{T_2 - T_3} \cdot (-T_3) \cdot (T_3 - T_2) \right\} \\
= & \left\{ m_I \cdot \mathbf{v}_I^{(1)} \cdot \delta \mathbf{x}_I^{(2)} - \delta \mathbf{x}_I^{(2)} \cdot \nabla^I V_{\text{pot}} \cdot T_2 \right\} + \left\{ -m_I \cdot \mathbf{v}_I^{(1)} \cdot \delta \mathbf{x}_I^{(2)} + \delta \mathbf{x}_I^{(2)} \cdot \nabla^I V_{\text{pot}} \cdot T_2 \right\} \;=\; 0
\end{aligned}
$$

vanishes exactly. Therefore the variation of Hamilton's quantity of action

$$
\delta S_{\text{Ham}} [\gamma] \;\stackrel{(122)}{>}\; 0 \qquad \forall \;\; T_2, \delta \mathbf{x}_I^{(2)}
$$

is positive definite for all basic 'three-step' variations $\text{RB}^{(1)} * \tilde{w}_1 * \text{RB}^{(2)} * \tilde{w}_2 * \text{RB}^{(3)} \left[T_2, \delta \mathbf{x}_I^{(2)} \right]$ - and hence for all Hamilton type variations $\delta \gamma^{(\text{Ham})}$ of free course γ of intrinsic action w.

□

Proposition 9 *Hamilton's quantity for variation (of free course γ) of - intrinsic - action w*

$$
0 \;<\; \delta S_{\text{Ham}} [\gamma] \;\stackrel{(114)}{=}\; - E_{\text{RB}^{(1)}} \cdot T_3 \;-\; E_{\text{RB}^{(2)}} \cdot (T_3 - T_2) \;-\; \nabla^I V_{\text{pot}} \cdot \delta \mathbf{x}_I^{(2)} \cdot T_3
$$

is determined by the - external - steering effort in terms of

1. *temporarily expended steering energy $-E_{\text{RB}^{(1)}}$ and $-E_{\text{RB}^{(2)}}$ into system $\{G_I\}$*
2. *corresponding activation duration $T_3 - T_1$ resp. $T_3 - T_2$ of the system until extraction at final steering moment T_3 and*
3. *passively coupled and extracted additional potential steering energy from temporary displacement along $\delta \mathbf{x}_I^{(2)}$.*

Proof: To generate *temporary* displacement $\delta \mathbf{x}_I^{(2)}$ from free evolution of intrinsic action w the engineer expends steering energy - from his external reservoir $\{ ①_{\mathbf{0}} \}$ - into system $\{G_I\}$

$$
-E_{\text{RB}^{(1)}} \;\stackrel{(117)(116)}{=}\; \frac{m_I}{2} \cdot \left(\frac{\delta \mathbf{x}_I^{(2)}}{T_2} \right)^2 \;+\; m_I \cdot \mathbf{v}_I^{(1)} \cdot \frac{\delta \mathbf{x}_I^{(2)}}{T_2}
$$

$$
-E_{\text{RB}^{(2)}} \;\stackrel{(121)}{=}\; - \frac{m_I}{2} \cdot \left(\frac{\delta \mathbf{x}_I^{(2)}}{T_2} \right)^2 - m_I \cdot \underbrace{\mathbf{v}_I^{(2)}}_{\stackrel{(115)}{=} \mathbf{v}_I^{(1)} - \frac{\nabla^I V_{\text{pot}}}{m_I} \cdot T_2} \cdot \frac{\delta \mathbf{x}_I^{(2)}}{T_2} \;+\; \frac{m_I}{2} \cdot \left(\frac{\delta \mathbf{x}_I^{(2)}}{T_2 - T_3} \right)^2 + m_I \cdot \underbrace{\mathbf{v}_I^{(2)}}_{\text{etc.}} \cdot \frac{\delta \mathbf{x}_I^{(2)}}{T_2 - T_3}
$$

which together with activation duration T_3 resp. $T_3 - T_2$ proves straightforward

$$
\begin{aligned}
& - E_{\text{RB}^{(1)}} \cdot T_3 \;-\; E_{\text{RB}^{(2)}} \cdot (T_3 - T_2) \\
& \qquad = \;\; \nabla^I V_{\text{pot}} \cdot \delta \mathbf{x}_I^{(2)} \cdot T_3 \;\; + \underbrace{\frac{m_I}{2} \cdot \left(\frac{\delta \mathbf{x}_I^{(2)}}{T_2} \right)^2 \cdot T_2 \;+\; \frac{m_I}{2} \cdot \left(\frac{\delta \mathbf{x}_I^{(2)}}{T_2 - T_3} \right)^2 \cdot (T_3 - T_2)}_{\stackrel{(122)}{=} \delta S_{\text{Ham}}[\gamma]} \quad .
\end{aligned}
$$

The system is activated with additional energy $-E_{\mathrm{RB}^{(1)}}$ and $-E_{\mathrm{RB}^{(2)}}$ for duration $T_3 - T_1$ resp. $T_3 - T_2$ until all extra energy is absorbed

$$E_{\mathrm{RB}^{(3)}} \stackrel{(120)(119)}{=} \frac{m_I}{2} \cdot \left(\frac{\delta \mathbf{x}_I^{(2)}}{T_2 - T_3} \right)^2 + m_I \cdot \underbrace{\mathbf{v}_I^{(3)}}_{\stackrel{(115)}{=} \mathbf{v}_I^{(1)} - \frac{\nabla^I V_{\mathrm{pot}}}{m_I} \cdot T_3} \cdot \frac{\delta \mathbf{x}_I^{(2)}}{T_2 - T_3}$$

again back into external calorimeter reservoir $\{①_{\mathbf{0}}\}$. The engineer retrieves all temporarily expended steering energy from system $\{G_I\}$ at final moment T_3 of his steering intervention

$$E_{\mathrm{RB}^{(1)}} + E_{\mathrm{RB}^{(2)}} + E_{\mathrm{RB}^{(3)}} \quad = \quad 0 \quad .$$

□

Corollary 6 *In free running - no exterior steering actions* $\mathrm{RB}^{(i)}$ *- course of intrinsic action* w *steering effort is absent. Hamilton's (positive definite) quantity of action is minimal.*

The variational analysis compares two physical processes - steered action and free evolving action. Steered actions are associated with extra steering effort. We leave analogous physical examination of the steering effort which generates Lagrange type variations $\delta\gamma^{(\mathrm{Lagr})}$ of the free course γ of intrinsic action w as a future exercise.

7 Discussion

The foundation of Dynamics is usually done axiomatically. That is good for purposes like didactical preparation and formal mathematical elegance. The whole formalism can be traced back to a manageable system of initial propositions which are logically independent from one another. Though this mathematical treatment of Physics already begins in the abstract. This obscures physical clearness (German: Anschaulichkeit) and stimulates questions for the physical meaning and the relation of this abstract mathematics to reality.[44] One can also look into what actually happens in measurement practice. We examine the origin of (basic) physical quantities, of quantitative equations and the genesis of algebra.

Planck [11] warns 'Activities from the *Axiomatiker* are useful and necessary but therein also hides the dubious danger of one-sidedness, that the physical world view loses its meaning and degenerates into empty formalism. Because if connection with reality is detached then a

[44]Poincarés [27] proclaims 'What science can grasp are not the things themselves but the relations between things'. Planck [11] distinguishes (i) the natural world - which we actually inhabit and of which we are a natural part of - (ii) our sensual perceptions and (iii) our physical conception of the world (German: physikalisches Weltbild). For Planck *physicists* are workers on the development of the physical world view. He classifies three groups depending on their main interest and method how they treat the theory: *Metaphysician*: stress its relation to real outside world. *Positivist*: stress its relation to sensual world. *Axiomatiker*: focus attention - neither on its relation to the real nor on those to the sensual world - but rather on inner coherence and logical structure of physical theory.

physical law *appears* - not anymore as relation between quantities which can all be measured independently from one another but - *as definition*, by means of which one of those quantities is reduced to the others.[45] Such *reinterpretation* is particularly tempting because a physical quantity can be defined much more exactly by an equation than by a *measurement*; however that fundamentally represents *abandonment of its true meaning.*'[46]

Our measurement theoretical conceptual order precedes a possible subsequent axiomatic reorganization of knowledge [7] [22]. An axiomatic formulation of Classical Mechanics starts with *mathematical terms* (mass m, force $\mathbf{F}$, momentum $\mathbf{p}$ etc.), it *postulates* their properties ($m = \text{const}$, $\mathbf{F}/\text{mod}\,\mathbf{v}$) and fundamental equations ($\mathbf{F} = m \cdot \mathbf{a}$, $\mathbf{p} = m \cdot \mathbf{v}$, $E_{\text{kin}} = \frac{m}{2} \cdot \mathbf{v}^2$). While the axiomatic system provides a good basis for deductive reasoning; the objective of our physical foundation is to justify these axioms themselves.[47] The mathematical theory *defines* derived quantities in terms of basic quantities and it *proves* equations in question from a manageable system of postulated fundamental propositions. This logical mathematical reasoning though starts from undefined basic elements and from unproven postulates.

Helmholtz [5] demands a further justification and derivation of axioms. The empiricist conception does not accept axioms (e.g. of geometry and arithmetic) as unprovable and not proof requiring propositions. Helmholtz distinguishes the act of counting and measuring. The physical meaning of 'quantity' and 'equal' is bound to the conduct of physical operations. Measurements have been scrutinized with regard to their *outcome*, information content about the measurement *object* and specification of the measurement *method* itself.[48] The result of a measurement - Mach [8] criticizes - does not provide absolute quantities. It is meaningful only with respect to a physical reference. In a measurement the outside world is not even directly accessible. Planck [11] emphasizes: *The two sentences - There is a real outside world. The real outside world is not directly accessible - are the crux of the matter of the whole natural science Physics.* Wallot [12] recognized the double nature of basic physical measures: the pair comparison of measurement object and of constructed material model by

[45]To concept formation of Mechanics (which is not simply realized by only formal operations with various possible relations alone) belongs aside from development of quantitative notions also the specification of their physical and methodical conditions. Ruben stresses 'A pure mathematical relation alone says nothing about a particular sensual concrete domain, thus has no *mechanical meaning* at all. The latter *comes only about if* symbols - which are connected in the mathematical relation - symbolize attributes which are concrete in experimental activities' [14].

[46]'What adds to the difficulty - Planck amends - is that maintaining the name (of a physical notion for derived expressions in the formalism) easily gives occasion to misunderstanding and obscurity' [11].

[47]To justify basic physical quantities we do not invoke to a new set of axioms - as Luce, Suppes in so called 'Theory of Measurement' [30]; a *representation theoretical conception* which grasps measurements as 'the assignment of numbers to objects and phenomena' and focusses on the (formal) analysis of 'their invariance under appropriate transformations' - we examine the practice of basic dynamical measurements itself.

[48]The international vocabulary of metrology [32] calls the result of a measurement 'quantity' and defines: *A 'quantity' is a property of a phenomenon, body, or substance, where the property has a magnitude that can be expressed as a number and a reference.* commonly symbolized: The 'quantity' equals $Q = \{Q\} \cdot [Q] + \Delta Q$ with unit measure or dimension $[Q]$, numerical value $\{Q\} := \sharp\,[Q]$ determined in the measurement method and sufficiently small measurement uncertainty ΔQ. The methodical question with regard to basic dynamical measures focusses on the origin of both the 'reference' and the operation for determining that 'number'.

means of which the former is specified (to sufficient precision).

In reception of Mach's historic and critical account of the evolution of Mechanics Hörz, Wollgast [5] demand that an explanation of known laws in scientific theories also requires the description of methods by means of which they were discovered. Abstract concepts are not given naturally. Development of new concepts is always connected with particular operations. The actual meaning of concepts is determined by those operations. The measurement theoretical foundation of physics plays a big role for ongoing relativization of absolute physical notions (as Einstein's critique of the conception of simultaneity already demonstrated) even today. In order to prevent the continuation of prejudices we always connect the conceptual critique with the inspection of operations which have lead to abstract concepts.[49]

7.1 Equivalence Relations

We refer to domains of everyday work experience where conditions for meaningful colloquial denominations 'material body' and 'motion' are practically sufficiently satisfied [15] [28]. On the basis of everyday work experience we roughly know what is meant by 'length', 'duration', 'potential to cause action', 'striking power' or 'impulse'. These colloquial notions refer to characterizations of relative motion between neighboring objects and to physical behavior in interactions of motion. Their precise physical meaning is defined by means of pre-theoretic *ordering relations* {2.3}:

- $\sim_l$ if two extended objects lie on top of each other: one will *cover* the other
- $\sim_t$ if two processes begin simultaneously: one will *outlast* the other
- $\sim_E$ coupled against same system $\{\mathcal{G}_I\}$: the effect of one source *exceeds* the other
- $\sim_{\mathbf{p}}$ in a collision against one another: one object *overruns* the other

In German the same circumstances are expressed with simple denominations (*überdecken*, *überdauern*, *übersteigen*, *überrennen*) which have a direct colloquial meaning.[50] Each comparison method is of a physical nature. They are universally reproducible in an observer independent way.

[49]The main objective for Mach's account of the historic evolution of mechanical principles [7] was 'to dispel metaphysics out of Physics... because we are used to call those notions metaphysical from which we have forgotten how we arrived at them.' It was also motivated pedagogically [8]: 'An insight is grasped best by the learner through the same way along which it was found.' The primary requirement in education Plank [11] explains: 'is not so much the amount of material but rather the way of its treatment... A single mathematical proposition which is truly understood by students has more value for them than ten memorized formulas... School is not supposed to convey technical routines but consequent methodical thinking.'

[50]Ruben zeigte daß die Bedeutung von Ausdrücken für elementare logische Operation zurückführt auf 'Bezeichnungen von Handlungen, die samt und sonders aus der Sprache des römischen Bauern stammen' [16]. Zur Klarstellung der Bedeutung von physikalischen Ausdrücken empfiehlt Ruben: Halte Dich nahe dran an der Muttersprache und achte genau darauf was Du damit eigentlich sagst.

Our measurement theoretical foundation of basic dynamical quantities is circularity free. We do not presuppose equations of motion and other mathematical relations between basic dynamical quantities (e.g. mathematical formulations of conservation laws, symmetries etc.). Every new basic measure: energy, momentum, inertial mass has to be explained in words or by examples because definition-*equations* for basic quantities do not exist [12].

7.2 Principles

From quantification of pre-theoretic ordering relations we obtain basic physical quantities. In a physical model - built by coupling congruent standard actions - energy and momentum become measurable. The construction of the physical model (for calorimetric measurements) relies on physical principles:

- *Principle of Causality*: under certain conditions something certain happens.
- *Principle of Inertia*: moving bodies move (without external agent) on their own. Motion is their way existence (German: Daseinsweise). We identify interactions of motion by changing state of motion.
- *Impossibility of a Perpetuum Mobile*: If coupled into a circular process both (reversible) processes are equivalent with regard to energy and momentum.
- *Principle of Sufficient Reason*: a reasonable external cause is responsible for the change in the state of motion of an object.
- *Equivalence Principle*: of intrinsic actions for relative boosts of system or observer. Under uniform boost and reorientation the *kinematical description* of respective states of motion transforms - active and passive - Galilei covariant (see Remark 8).
- *Superposition Principle*: of compatible intrinsic actions w and external steering resp. measurement actions $w_{\mathbf{1}}$ and W_{cal}

and on principles of methodical nature:

- *Basic measurement*: as doubling of physical measures (see Remark 4). The act of basic measurement is a pair comparison between measurement object and material model.
- *Congruence Principle*: for reliable quantification we count congruent dynamical units. In physical model for Energy, Momentum and Inertial Mass we count the number of equivalent elements $\mathbf{1}_E\big|_0$, $\mathbf{1}_\mathbf{p}$ resp. ① (see Theorem 2).
- *Equipollence Principle*: of measuring the cause of potential action by its (kinetic) effect. Provided Impossibility of a Perpetuum Mobile this implies *Conservation of Energy.*

We begin from the action as an inseparable unit and from the method of basic measurements. We quantify intrinsic action w by means of unit actions $w_{\mathbf{1}}$. $\mathcal{A}$lice may pick out a *standard* from the variety of possible interactions of motions which is reproducible and available anywhere and anytime and in any number. We presuppose both actions solely as completed processes with regard to changes in their final state of motion. Taken by themselves - as inseparable measurement *unit* $w_{\mathbf{1}}$ - they are also *unquantified* but these units are *congruent* among one another (see Remark 3). In Galilei Kinematics we construct a material model W_{cal} (for a calorimetric measurement) which solely consists of congruent unit actions $w_{\mathbf{1}}$. We set up and couple a sequence of congruent actions $w_{\mathbf{1}} * \ldots * w_{\mathbf{1}}$ by physical operations: Consecutive actions are associated in standardized object ① which in between each action moves freely. The course of their couplings is controlled from the outside by physicists (see figure 7, 13). Hence for individual unit action $w_{\mathbf{1}}$ it only matters which change in the state of motion is ultimately attained - irrespective of details in its spatiotemporal progression. A team of physicists couples dynamical units from an external reservoir $\left\{ \mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}}, ①_{\mathbf{0}} \right\}$. They steer initiation of each individual action such that the desired effect is achieved. Everybody has to know *when* and *where* to pick the next initially resting object ① from the calorimeter reservoir $\{①_{\mathbf{0}}\}$ and *how* to catapult it into the way of incident particle $ⓐ_{\mathbf{v}_a}$ to generate impulse reversion or absorption. Physicists have to cooperate to construct material models. Basic physical quantities are a joint product and not generated individually.

In this model we can count the number of equivalent dynamical units $\mathbf{1}_E\big|_0$ and $\mathbf{1}_{\mathbf{p}}$ which get extracted from calorimeter reservoir $\{①_{\mathbf{0}}\}$. In this way pre-theoretic notions Energy, Momentum and Inertial Mass become measurable. The congruence principle is constitutive for basic physical quantities. We justify the origin of (basic) physical quantities from principles of empirical practice.

7.3 Genesis of Algebra

The essence of arithmetic laws - Mach [8] explains - is to introduce novel ways of 'mediated counting'. Each arithmetic law establishes an abbreviating relation between two ways - the laborious direct and the elegant indirect - of counting. The result can always be deduced in the more complicated direct way from the definition of elementary counting. In a *mathematical proof* we essentially reduce complex relations between derived terms (sums, products, integrals etc.) to the relation $1 = 1$ of basic elements (numbers).

Similarly we generate fundamental equations between basic physical quantities. Our *physical proof* is based on physical principles and constructing material models. Arithmetic addition '+' of (extensive) energy and momentum quantities is a genetic consequence of underlying physical operations. We associate energy and momentum of two elements $ⓐ_{\mathbf{v}_a}$ and $ⓑ_{\mathbf{v}_b}$ by absorbing them in same calorimeter $\{①_{\mathbf{0}}\}$. Adding '+' quantities of energy and momentum $(E, \mathbf{p})_a + (E, \mathbf{p})_b$ corresponds to basic counting. We count the combined number of extractable dynamical units $\mathrm{RB}[ⓐ_{\mathbf{v}_a} \Rightarrow ⓐ_{\mathbf{0}}] + \mathrm{RB}[ⓑ_{\mathbf{v}_b} \Rightarrow ⓑ_{\mathbf{0}}]$ from absorbing both elements $ⓐ_{\mathbf{v}_a}$ and $ⓑ_{\mathbf{v}_b}$ in same calorimeter reservoir $\{①_{\mathbf{0}}\}$.

Calorimeter model W_{cal} constitutes a relation between basic dynamical measures. $\mathcal{A}$lice can conduct basic measurements for inertial mass $m\,[ⓐ] =: m_a^{(\mathcal{A})} \cdot m_{\mathbf{1}^{(\mathcal{A})}}$, kinetic energy $E\,[ⓐ_{\mathbf{v}_a}] =: E_a^{(\mathcal{A})} \cdot E_{\mathbf{1}^{(\mathcal{A})}}$ and momentum $\mathbf{p}\,[ⓐ_{\mathbf{v}_a}] =: \mathbf{p}_a^{(\mathcal{A})} \cdot \mathbf{p}_{\mathbf{1}^{(\mathcal{A})}}$. Each is - independently - quantified by the number $m_a^{(\mathcal{A})}, E_a^{(\mathcal{A})}$ resp. $\mathbf{p}_a^{(\mathcal{A})}$ of equivalent elements in her physical model ①, $\mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}}$ resp. $①_{\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}}}$ and the unit measure which each of them represents $m_{\mathbf{1}^{(\mathcal{A})}}, E_{\mathbf{1}^{(\mathcal{A})}}, \mathbf{p}_{\mathbf{1}^{(\mathcal{A})}}$. The relation between $\mathcal{A}$lice physical quantities $m_a^{(\mathcal{A})}, E_a^{(\mathcal{A})}$ and $\mathbf{p}_a^{(\mathcal{A})}$ follows from interrelation of respective unit objects ①, energetic units $\mathbf{1}_E^{(\mathcal{A})}\big|_{\mathbf{0}}$ and impulse units $①_{\mathbf{v}_{\mathbf{1}^{(\mathcal{A})}}}$ in calorimeter model W_{cal}. We *justify* equations between physical quantities of energy and momentum from the design of underlying calorimeter-collision-cascade $W_{\text{cal}} := w_{\mathbf{1}} * \ldots * w_{\mathbf{1}}$. If $\mathcal{A}$lice constructs the calorimeter model in Galilei-Kinematics then her measured values for kinetic energy $E_a^{(\mathcal{A})}$, impulse $\mathbf{p}_a^{(\mathcal{A})}$, inertial mass $m_a^{(\mathcal{A})}$ and velocity $\mathbf{v}_a^{(\mathcal{A})}$ satisfy quantitative equations $E_{\text{kin}} = \frac{m}{2} \cdot \mathbf{v}^2$, $\mathbf{p} = m \cdot \mathbf{v}$ in which numerical values occur in the form *measure/unit measure* $E_a^{(\mathcal{A})} := \frac{E[ⓐ\ \mathbf{v}_a]}{E_{\mathbf{1}^{(\mathcal{A})}}}$, $p_a^{(\mathcal{A})} := \frac{p[ⓐ\ \mathbf{v}_a]}{p_{\mathbf{1}^{(\mathcal{A})}}}$, $m_a^{(\mathcal{A})} := \frac{m[ⓐ]}{m_{\mathbf{1}^{(\mathcal{A})}}}$ and $v_a^{(\mathcal{A})} := \frac{v_a}{v_{\mathbf{1}^{(\mathcal{A})}}}$. We take into account all aspects of measurement practice.[51] We can perceive the validity of equivalence relation $\sim_{E,\mathbf{p}}$ in practice and carry out the concatenation $*$ manually. 'In Gedanken' we can carry out actual measurements of energy and momentum in a fundamental manner and thus proof fundamental equations of (classical and relativistic) Dynamics based on physical and methodical principles.

Analysis of spatiotemporal evolution is linked to practical steering effort. In a physical system we can not turn off the interaction between its elements G_i. Though we can do the contrary, we can include additional actions. Physicists - temporarily - couple their calorimeter into system $G_1 \cup \ldots \cup G_N$ to prepare the initial state of motion for individual elements and steer the course of intrinsic action w: By controlled association of external steering actions $\text{RB}^{(i)}$ and consecutive segments of intrinsic action w_i physicists can control the course of intrinsic action w and analyze corresponding steering effort. Provided *basic* physical quantities length, duration and energy, momentum we introduce more differentiated terminology suitable for analyzing continuous evolution. We define 'force' of intrinsic action w as a meaningful *derived* physical quantity $\mathbf{F}_a^{(\mathcal{A})} := \frac{\Delta \mathbf{p}_a^{(\mathcal{A})}}{\Delta t_a^{(\mathcal{A})}} \left[w\big|_{\mathbf{x}_I, \mathbf{v}_I} \right] /\text{mod}\ \mathbf{v}_I$ and 'displace-

[51] A major hindrance to the understanding of fundamental measurements in Dynamics has been the failure to uncover suitable empirical concatenation operations $*$ for attributes 'potential to cause actions' and 'striking power' or 'impulse'. Luce, Suppes et al define the 'attributes kinetic energy and momentum having two independent components: mass and velocity... In conjoint-measurement theory, the way in which each component affects the kinetic energy resp. momentum is studied by discovering which changes must be made in one component to compensate changes in the other' [30]. They can *verify* quantitative equations $E_{\text{kin}} = \frac{m}{2} \cdot \mathbf{v}^2$, $\mathbf{p} = m \cdot \mathbf{v}$ in individual cases. But they *cannot prove general validity* (and limitation) of these equations because they do not take underlying physical and methodical principles into consideration.

Luce, Suppes conception of 'Theory of Measurement' *primarily focusses on numerical representations* of pre-theoretic ordering relations (i.e. of empirical relational system onto numbers) *and uniqueness theorems*. They presuppose abstract mathematics (representation spaces etc.) as given and characterize 'measurement statements as empirically meaningful only if its truth value is invariant under the appropriate transformations of the numerical quantities involved' [29]. Their representation theoretical conception is *complementary* to our *physical perspective*: (i) empirical meaningfulness is assured by specifying measurement operations which are universally reproducible in an observer independent way and (ii) Physics turns out as mother of its Mathematics in empirical practice - without presupposing it.

ment work' in steered actions.[52] We determine properties and interrelations of these derived physical quantities. From the differentiated specification of momentum evolution we *induce* Newton's *equations* for the evolution *of motion*.

We examine the variational analysis from a physical perspective. The variation $\delta\gamma^{(\mathrm{Ham})}$ of the free course γ of intrinsic action w involves an active physicist as well. We examine the physical process behind basic variation maneuvers, the coupling of extrinsic steering actions $\mathrm{RB}^{(1)} * w_1 * \mathrm{RB}^{(2)} * w_2 * \ldots$ into the course of intrinsic action w. For the variational analysis (of the steering effort) we compare two physical processes - steered action and free evolving action. For every local Hamilton type variation $\delta\gamma^{(\mathrm{Ham})}$ of the free evolution γ of intrinsic action w the variation of Hamilton's quantity of action is positive definite $0 < \delta S_{\mathrm{Ham}}[\gamma] \quad \forall\, \delta\gamma^{(\mathrm{Ham})}$. Steered actions are associated with extra steering effort.

Step by step physical terminology is constructed from empirical grounds.[53] We reconstruct the formation of the theory. In the resulting formalism mathematical terms inherit physical meaning and their interrelation genetically unfolds. Each step in the development of new physical concepts is the product of a working process (see figure 24). We grasp the scientific work of physicists - their generation of new abstract notions - as production process.[54] One can understand the origin of basic physical quantities from an empirical basis

- without any mathematical presuppositions
- active role of (a collective of) physicists and
- genesis of the mathematical formulation.

On one side is the *empirical practice* and on the other the *birth of mathematical formalism*.

[52]Newton's force used to be the main and basic concept of Mechanics. In more recent development of Physics - Planck [11] states - 'Newton force has lost its fundamental importance for theoretical physics. In the modern layout of Mechanics it only appears as secondary quantity, replaced by another higher and broader notion of work and potential energy.' Just what measurement theoretical analysis yields directly!

[53]Despite the Nobel - Laughlin [35] 'focussed on the question whether Physics was a logical creation of the mind or a synthesis built on observations... Seeing our understanding of nature as a mathematical construction has fundamentally different implications from seeing it as an empirical synthesis... (from a) world view that mathematics grows out of experimental observation, not the other way around. The world we actually inhabit, as opposed to the happy idealization of modern scientific mythology, is filled with wonderful and important things we have not yet seen because we have not looked, or have not been able to look at due to technical limitations.' 'It is not the case that in natural science, as in any other science, we start from fixed basic concepts and search for their realization in our surrounding world - explains Planck [11] - rather it is quite the reverse. By birth we humans are all simply placed into the midst of life without prior preparation even without being informed and in order to cope with this imposed life... we seek to *establish* certain *concepts which are suitable for usage* to past and future experiences *in real life*.'

[54]Ruben [19] examines the reflection of practical activities in the activities of consciousness: 'The products of consciousness are linked with practical activities. For Hegel the natural scientist is producer of knowledge, not an owner of knowledge which exists independently of the action of scientists per se. Hegel assumes the theoretical doing itself as work. 'Thinking as work' - that is the great basic idea of Hegel's philosophy (German: Wissenschaft als Arbeit). Under the expression 'work' Hegel understands both that an active subject finds an object which he reshapes for his purpose (thus converts into something different to what it has been before) and also that in this doing the subject materializes his own ability in the other object.'

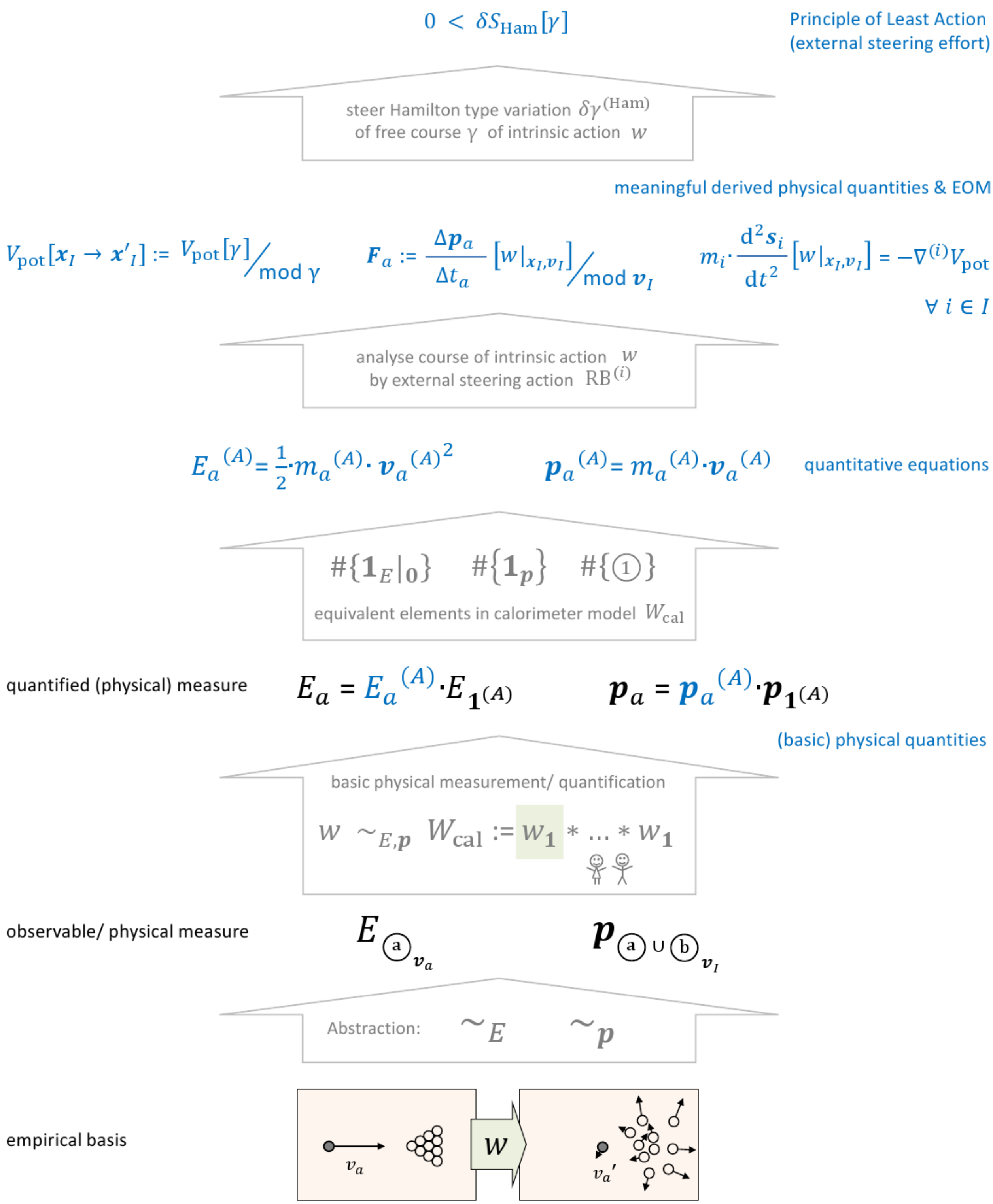

Figure 24: products of a (practical and theoretical) working process

The measurement theoretical foundation provides the *mathematical formulation* of abstract measurement results together with its *physical conditions*. From the *interrelation* of conditions of measurement actions (steered impulse reversion and calorimetric absorption) the mathematical principles of Newton's axiomatic system are justified. Under modified conditions of Poincare Kinematics the limitations of Newton's mathematical formalism becomes transparent. Same measurement and steering actions lead to basic physical quantities (energy and momentum) of relativistic dynamics [25]. In this way apparently problematic definition of derived physical quantities (inertial mass and force [33]) is physically resolved.

We demonstrate the possibility to generate basic dynamical notions from physical and methodical principles. They concern the formation of basic physical quantities energy and momentum in measurement practice - and thus the basis for all derived terminology of Dynamics {5} {6}. This possible starting point differs from formal axiomatic approaches insofar as contrary assumption of these principles would render impossible a meaningful and reproducible physical science of nature. These basic measurement principles are therefore indispensable prerequisites for Physics! At the same time they bring forward clearness and conditions of physical quantities. We demonstrate the benefit of this combined *way of thinking* in Physics. Domain of validity and limitation of fundamental equations - in Classical, Analytical and Relativistic Mechanics - becomes transparent.

The existence of meaningful physical quantities for Dynamics {3} - and derived from them fundamental equations {5} {6} - is tied to conditions (see figure 24). All elements of the model (congruent unit actions $w_{\mathbf{1}}$) must satisfy physical conditions and the construction procedure for the model (their controlled association '$*$' for impulse reversion $W_{(i)}$ (29) and for absorption W_{cal} (42)) must be realizable. Then we arrive - by counting equivalent dynamical units $\mathbf{1}_E\big|_0$ and $\mathbf{1}_{\mathbf{p}}$ in material model W_{cal} - at meaningful basic physical quantities of energy and momentum. One can check whether those conditions are satisfied for gravitational and quantum mechanical circumstances and to what extent one wants to use the corresponding mathematical formulation.[55]

When we retrospect what we actually have done, we notice with Ruben [15] that mathematics is not only applied in physics. In reverse Physics also appears as the mother of (its) Mathematics in empirical practice. We begin from pre-theoretic ordering relations 'more impulse' and 'more energetic' and from the method of basic measurements. In a physical model - built by coupling congruent unit actions $w_{\mathbf{1}}$ - those pre-theoretic notions become measurable. We derive all equations between basic physical quantities of Energy, Momentum and Inertial Mass and ultimately the Principle of Least Action. In retrospect pure mathematics appears like something half [36]. We have completed physics with the practical. Now we know more than before. This work is a contribution to understanding the active role of a physicist in basic measurements.

[55] The measurement theoretical view points out limitations of familiar approach but also from what grounds a mathematical formulation - adapted to quantum mechanical and gravitational conditions - can arise.

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