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On different ways of being equal

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Abstract The aim of this paper is to present a constructive solution to Frege's puzzle (largely limited to the mathematical context) based on type theory. Two ways in which an equality statement may be said to have cognitive significance are distinguished. One concerns the mode of presentation of the equality, the other its mode of proof. Frege's distinction between sense and reference, which emphasizes the former aspect, cannot adequately explain the cognitive significance of equality statements unless a clear identity criterion for senses is provided. It is argued that providing a solution based on proofs is more satisfactory from the standpoint of constructive semantics.

1 Introduction

There is hardly any doubt that some mathematical equality statements carry a real piece of information while others contain no epistemic content whatsoever. One striking aspect of this curious phenomenon is well known since the work of Frege (1892):

- Objects on both sides of an equality statement may be presented in different ways. Recognizing objects as the same again when they are given to us differently is surely not a simple matter in mathematics. Compare, for example, the trivial equality $9391536 = 9391536$ with the prime factorization $2^4 \cdot 3^2 \cdot 7^2 \cdot 11^3 = 9391536$, a non-self-evident truth that can only be known after a large amount of calculation. Why is this equality not obvious if both objects are the same?

Frege's theory of cognitive significance (*Erkenntniswert*) attempts to explain what makes some equality statements have epistemic value by means of a distinction between the sense and reference of an expression. More concretely, it says that when we apprehend the truth of an equality claim relating objects that are given to us in a different way, we are capable of extending our current knowledge by drawing new inferences about the different modes of presentation (senses) of those objects (referents). To give an example, the knowledge that $2^4 \cdot 3^2 \cdot 7^2 \cdot 11^3 = 9391536$ is the case allows us to realize that the number 9391536 is divisible by 49 straight away, without having to do too much calculation. By the same token, some equality statements seem less revealing because they relate two objects with the same mode of presentation or sense. Equalities that must hold by definition like $1 = 0.\overline{9}$ appear to fall into this category, for one can hardly say that their knowledge produces a gain of information. Even though 1 and $0.\overline{9}$ are entirely different as expressions, they have the same sense.

The main goal of this paper is to call attention to a second aspect of Frege's puzzle, a new dimension which so far has received little consideration, but offers a brand new perspective to the understanding of cognitive significance. It concerns not the "mode of presentation" but what we call the "mode of proof" of an equality:

– Substantial arguments might be required to support a claim that two objects are one and the same. Proving an equality statement is often a difficult task in everyday mathematics. Why is this so hard if the objects are just the same? This is most likely because elaborate arguments are needed in the proof. Unlike $-1 = -1$, which is a straightforward consequence of the reflexivity of equality, some mathematical maturity is required to prove that $e^{i\pi} = -1$ is true.

Those are really just two facets of the same problem: on the one hand, difference in mode of presentation causes an equality to require a sophisticated proof, and, on the other hand, if non-trivial arguments are necessary to make an equality proof go through then clearly the mode of presentation of both objects cannot be the same.

When talking about proofs it is natural to think of the constructive conception of mathematics, a longstanding tradition established by Brouwer, Heyting, Dummett, Prawitz, Martin-Löf, and Sundholm, among others, where a notion of "construction" that is similar but subtly different to that of a proof is given a prominent role in the determination of the meaning of the logical constants and the truth of a proposition. Indeed, the starting point of this paper is this very constructive semantics, or, more specifically, a correspondence between constructive semantics and type theory given via the meaning explanations of Martin-Löf (1982). There, a construction represents an untyped computation which is generally accepted as a correct step-by-step specification of how to obtain a certain mathematical object, an idea that is perhaps best expressed by Bishop's (1967) notion of a "person program". Based on those assumptions regarding the correctness of the meaning explanations and their role in understanding constructive mathematics, this paper proposes a solution to Frege's puzzle using type theory. As we shall see, it is no coincidence that we favor the "mode of proof" aspect of the puzzle: just as any other concept in mathematical constructivism, the cognitive significance of an equality statement is to be based on the fundamental notions of construction and proof.

The rest of the paper is structured as follows: First, a review of the problem of cognitive significance and Frege's theory of sense and reference is given in Section 2. Then, a brief articulation of constructive semantics is provided in Section 3. Next, we discuss type theory and compare some aspects of its two traditional flavors, which contrast an extensional and intensional interpretation of equality. We claim that the extensional version is a good approximation to the principles of constructive semantics, so it may closely represent cognitive significance from this perspective. This is the subject of Section 4, which is the central section of the paper, for it also contains our own theory of cognitive significance.

Section 5 is concerned with the sense/reference distinction within constructive semantics. Alternative homotopy type-theoretic accounts of cognitive significance are considered in Section 6. Some features of our account, related work, and possible objections are discussed in Section 7. Finally, Section 8 provides our conclusions.

2 The problem of cognitive significance

The problem of cognitive significance known as Frege's puzzle is the problem of explaining the informativeness of mathematical equality statements: if equal means "exactly the same as", then all theorems of the form $a = b$ would collapse into an obvious truth $a = a$ and all mathematical equations would be trivially true.[1] The puzzle is so called because it is a recurrent topic in Frege's writings. In particular, Frege's concerns are clearly expressed in a letter to Peano, in which he writes the following about the view that equality means exactly the same:

> What stands on the way of a general acceptance of this view is frequently the following objection: it is thought that the whole content of arithmetic would then reduce to the principle of identity, $a = a$, and that there would be nothing more than boring instances of this boring principle. If this were true, mathematics would indeed have a very strange content. (Frege, 1982a, p. 126)

It is difficult to say what is, according to Frege, the nature of the things that may have cognitive significance, for he never gave an explicit account of the subject. Perhaps a better question is whether we should attribute cognitive significance to the things that the rules of inference operate on (judgments) or the things that the logical connectives operate on (propositions). Unlike propositions, judgments have assertive force and may state that one or more objects have a property or stand in a relation. When one asserts a particular judgment one expresses the knowledge that what is being judged is indeed the case. It is thus common to regard judgments as objects of knowledge (Martin-Löf, 1985).

Since the works of Bolzano (1837) and Frege (1879), modern logic has been based on one basic form of judgment, namely, the judgment form A true which states that a proposition is true. From passages such as the following one, it appears that Frege chooses to attribute cognitive significance to true propositions, that is, judgments:[2]

> the cognitive value of $a = a$ becomes essentially equal to that of $a = b$, provided $a = b$ is true (Frege, 1892)

In any event, Frege (1892) seems more concerned about the conditions under which a true equality should have cognitive significance, which, according to him, is when it may contribute to one's advance of knowledge. One way of refining this idea is to propose the following adequacy criterion:

[1] See for instance Corazza and Dokič (1995); May (2001).

[2] False equality propositions have cognitive significance in our account, see Section 7.

Definition 2.1 (Adequacy)**.** *A theory of cognitive significance is adequate if it ensures that an equality statement has cognitive significance provided that a rational subject who gets to know the truth of this statement is justified in extending their knowledge.*

Therefore, a rational subject is justified to extend their knowledge with the assertion of an equality statement if the knowledge that this statement is true allows them to draw new inferences, as in Corazza and Dokič (1995). In modern logic which only has one form of judgment, this means that a subject who knows $a = b$ true and gets to know $P(a)$ true is well-justified in extending their knowledge by deducing $P(b)$ true. Since we shall be dealing with type theory later on, it is useful to bear in mind that this characterization of extension of knowledge can be applied to other forms of judgment as well.

2.1 Frege's theory of sense and reference

Frege's (1892) well-known solution to the puzzle that bears his name is to introduce a threefold division between expression, sense and reference. Expressions are either singular terms or sentences. The reference of a singular term is the object it stands for and its sense the mode of presentation of the reference. The theory is classical: the reference of a sentence is a truth value and its sense the truth condition expressed, which is typically identified with the proposition expressed by the sentence. Different senses may correspond to the same reference, but in an exact science, each sentence must determine exactly one truth value (i.e. either true or false), so every sentence in mathematics should respect the law of excluded middle.[3]

With a clear separation between sense and reference, Frege is able to explain that the cognitive significance of an equality statement comes from the difference of the senses of the expressions standing in both sides of the equality sign. One might think that his theory of sense and reference is therefore adequate: if 'a' and 'b' have different senses, then, as the sense of a sentence is the proposition it expresses, '$P(a)$' expresses a different sense from '$P(b)$', for some predicate P. Thus, assuming that 'a' and 'b' have different senses, one is immediately able to infer, from the knowledge that $a = b$ and $P(a)$ are the case, that a new proposition $P(b)$ holds. Unfortunately, this is not quite true. Even in a limited scope such as mathematics, Frege's theory of sense and reference has problems of its own, for it does not specify what it is for two expressions to have different senses.

Frege's solution to the puzzle presupposes a clear specification of what sort of identity criteria are applicable for senses, and he does make a strong case for the importance of equality of sense, as can be seen in passages such as:

> It seems to me that an objective criterion is necessary for recognizing a thought again as the same, for without it logical analysis is impossible. (Frege, 1982b, p. 70)

[3]See *Grundgesetze* (Frege, 1962, II, §57).

Yet, Frege (1962) is surprisingly indifferent about the conditions under which two singular terms express the same sense in *Grundgesetze*, the book in which he intends to present his ultimate foundational theory for arithmetic. At the same time, if a cognitively significant true equality statement is based on the difference of the senses expressed by the terms it is made up with, then providing a sharp identity criterion for senses is the most crucial step for the solution of the puzzle.[4]

Sense is only briefly explained in *Grundgesetze* (Frege, 1962, I, §2,§32), and there is no mention of sameness. But since Frege has proposed at least three identity criteria for senses elsewhere, we have the following candidates:

1. equality up to logical equivalence;
2. equality up to equipollence;
3. equality up to different choices of bound variables.[5]

As all mathematical theorems are obviously logically equivalent, the first criterion, which is proposed in a letter to Husserl (Frege, 1982b, pp. 70–71), is surely incompatible with the sense/reference solution to Frege's puzzle, so we have no option but to set it aside.

Frege's second suggestion is given by means of his equipollence principle, which states that two sentences A and B express the same sense just in case anyone who accepts A as true is justified in accepting B as true and vice-versa:

> Now two propositions A and B can stand in such a relation that anyone who recognizes the content of A as true must thereby also recognize the content of B as true and, conversely [...] So one has to separate off from the content of a proposition the part that alone can be accepted as true or rejected as false. I call this part the thought expressed by the proposition. (Frege, 1906, pp.197–98)

[4] The possibility should be considered that Frege did not think that providing an identity criterion for senses in *Grundgesetze* was important. Ruffino (1997), Duarte (2009), and Klement (2016) observed that the thesis that the reference of a sentence is a truth value was absolutely necessary to introduce Axiom IV to the system in the book. This axiom is crucial to prove that $(A \supset B) \supset (B \supset A) \supset (A = B)$, a theorem without which Frege could not show that the number of the set A equals the number of the set B iff there is a bijection from A to B (Hume's principle). But as long as the sense and reference of an expression are not kept apart, there may be false instances of it. For example, $(1 + 1 = 2) \supset (2 = 2) \supset ((1 + 1 = 2) = (2 = 2))$ seems to be a valid instance for references, but it is false for senses according to Frege (1891). As Ruffino (1997) suggests, Frege's sense/reference distinction may have been motivated by purely mathematical reasons, a necessary change in order to make the technical development of his logicism plausible.

[5] Klement (2002, 2016) also distinguishes three other identity criteria for senses: (4) an intermediate interpretation where the identity criterion for senses are stipulated by abstraction principles such as Frege's infamous Axiom V and (5) a fine view according to which the sense of a closed term has as its parts only the senses of the primitive expressions forming it. In either case, Klement (2016, §5) acknowledges that it is hard to amend the theory of sense and reference of *Grundgesetze* with these interpretations, given that all these identity criteria for senses give rise to problems on their own.

But because of its subjective appeal, the principle seems to go against Frege's anti-psychologist view of logic, as it is stated in terms of the psychological attitudes an individual may potentially have towards the acceptance of the truth of a sentence. Klement (2002, p.90) remarks that equipollence can be understood objectively using Leibniz Law, that is, two sentences A and B express the same sense just in case A and B can be substituted for each other in all (ordinary or oblique) contexts without change of truth-value.

Frege has also at least once explicitly said that two sentences express the same sense if they only differ in the choice of bound variables, which suggests that he had a conception of α-conversion for senses, a notorious equivalence in the lambda calculus.

> If we just had '$x^2 - 4x$' we could write instead '$y^2 - 4y$' without altering the sense; for 'y' like 'x' indicates a number only indefinitely. (Frege, 1891, p.11)

It is not easy to say whether Frege thought that α-equivalence *determines* equality of sense. However, as it provides a fine identity criterion that conforms to the equipollence principle described above, it seems a quite compelling alternative.

3 Constructive semantics and type systems

Either classically or constructively, a claim that a proposition is true is justified by means of a proof. Put differently, proofs are given for judgments. What is specific to the constructive conception is the basic idea that a proposition is true if there is a construction of it. In this context, constructions are given for propositions.

It is clear that, on pain of circularity, a construction of a proposition cannot be the same thing as the proof of a judgment, otherwise the assertion of a judgment of the form A true would not lead to a gain of knowledge. The reason judgments are regarded as objects of knowledge is because their assertion reflects real knowledge. Constructively, when one asserts a judgment of the form A true, that is, when one gives a proof of that judgment, one gets to know a certain construction that realizes the truth of A. But there is no reason to assume that the construction of the proposition A obtained is the proof of the judgment A true.

This subtle but crucial point should mark the distinction between the formalist and constructivist standpoints, for the formalist believes that there is nothing beyond a proof that a proposition is true (except for the proof itself), while the constructivist holds that it gives us a construction of that proposition.[6] Of course, Brouwer (1954) strongly criticizes the formalist position. For Brouwer, constructions are mental processes given in intuition, free creations of the mind. The idea of a totality of all constructions is unfounded, for the range of all possibilities is open-ended (Brouwer, 1907, pp.148-149). But the prevalence of elements

[6] Yet, this circular identification of constructibility with provability is now widespread in the literature. It can be traced back to Dummett's (1978) influential reconstruction of the intuitionism of Brouwer and Heyting in terms of proof as the primitive notion, which is a result of his radical rejection of the traditional account of constructions as free mental processes given in intuition.

of mysticism in Brouwer's writings precludes a precise understanding of his philosophy. Heyting (1931) does a better job at connecting constructions to intuition. For even though there is almost no explicit reference to intuition in his work, his explanation of constructions as fulfillments of intentions can be interpreted in light of Husserl's theory of intentionality, as already observed by Martin-Löf (1985) and Tieszen (1995). That is because, in Husserl's writings, intuition is understood in terms of fulfillment of intentions.

Recently, Martin-Löf (1982) proposed an alternative account of constructions that is purely computational but can nevertheless be articulated in a similar vein. It views constructions as programs given in a primitive (untyped) notion of computation. If we think of the ability of performing computation as a product of our human faculties, such as a form of intuition of computation, then it is perhaps fair to say that those constructions are free mental creations as well, in the sense of Brouwer (1907). From this standpoint, constructivism can be articulated as an informal semantics that interprets propositions as program specifications and then assign constructions to propositions based on the values they compute to. In addition to that, this intuitive semantics, called the "meaning explanations", is used to provide a justification for the rules of inference of type theory—a family of formal systems based on Russell's idea of annotating objects with types and restricting operations to objects of certain types.

The moral of the story is that type theory can serve as a language to talk about the basic concepts of mathematical constructivism: we view types as propositions and the elements of a type (called terms) as constructions that realize that proposition (Martin-Löf, 1982). Naturally, this means that terms are programs and types are program specifications. There are actually many different versions of type theory in existence, but their most relevant divergences can be succinctly described as a dichotomy between intensional and extensional approaches to type theory. Both accounts give a distinct portrayal of constructive semantics via the meaning explanations, depending on how close their formal system mirrors the intended interpretation. It goes without saying that both portraits are necessarily incomplete given Gödel's first incompleteness theorem. The question is which one provides a more faithful representation of constructivism in the sense above.[7]

4 Equality in type theory

Equality is one of the most controversial topics in type theory. This disagreement appears to be the result of a much deeper discussion concerning the questions of what should count as a proof and what a term of a type should represent.

In intensional type theory (Martin-Löf, 1975), terms act as formal representations of proof trees in a given formal system and their types serve as the corresponding judgments which are being proved. Typically the formal system in question is inspired by a constructive theory

[7]This is certainly not a trivial question, considering that Martin-Löf Martin-Löf (1975, 1982), the originator of both intensional and extensional traditions in type theory, has vacillated between regarding one approach as most fitting with the meaning explanations—until he finally decided adopting intensional type theory as the basis of his philosophical investigations in the late 80s. It appears this decision is inspired by a shift of viewpoint regarding the nature of constructions, which begin to be explicitly treated as proofs (Martin-Löf, 1987). But, as I mentioned in Section 3, the failure to distinguish proof and construction can be quite problematic.

of arithmetic such as Heyting arithmetic, and, consequently, the type theory must make sure that if a is a derivation of a judgment A true in that system, then a is a term of type A. This is the syntactic correspondence observed by Howard (1980), which is better called "judgments-as-types" because it is not about the constructions (programs) that realize a proposition in the sense of constructive semantics discussed in the previous section, but instead about the formal proofs of a judgment in a fixed formal system. One of the key aspects of mathematical constructivism is the idea that constructibility determines truth (Martin-Löf, 1985) and, again, given Gödel's incompleteness results, it is not surprising to see that constructions cannot be identified with formal proofs, as the judgments-as-types correspondence seems to imply.

Extensional type theory, on the other hand, is a formal approximation of the view of constructions as computations. In this version of type theory, the terms of a type do not stand for formal proofs but programs which realize the truth of the proposition that is being represented by that type. It is not intended as a theory of proof but as a theory of computation: terms are programs that may not even terminate. By contrast with the intensional approach, it does not make sense to regard terms as proofs here because, just as realizers serve no proof-theoretic purpose, terms need not convey information about the argument used to show that they are indeed well-typed. That is why the typing relation $a : A$ (pronounced "a is a term of A") is usually expected to be decidable in intensional type theory, since we should be able to recognize a proof when we see one, as suggested by Kreisel (1962). But this is merely an idealization of the notion of construction. If one is to fully adhere to the view of construction as a free product of a primitive "intuition" of computation, then one should also take constructions that may run forever into account and there is no reason to suppose that there exists a procedure for deciding when a construction realizes a proposition (See Subsection 4.1). In extensional type theory, this is reflected by the fact that the typing relation is generally undecidable. Only the proof of a typing judgment counts as evidence, never the term itself, much like the distinction between proof and construction in constructive semantics.

Ultimately, constructive semantics justify both the intensional and extensional flavors of type theory, so perhaps the choice of one over another is philosophically inessential after all. But, even so, it is not unreasonable to ask whether one kind of formalism is more complete with respect to the intended interpretation than the other, meaning that it reflects a picture that is closer to the truth, so to speak.

4.1 Intensional versus extensional equality

Both intensional and extensional type theory are based on the same notion of a type, so it is appropriate to begin the discussion with a general account of what is a type. In order to prescribe a type one has to specify

(i) how to construct the canonical terms of that type;

(ii) how to show that two canonical terms of that type are equal.

Note that this is supposed to reflect the quintessential idea in constructive semantics that to know what a proposition is is to know how to obtain a construction of it (Martin-Löf, 1985), except that it adds an additional requirement that we should also know when two constructions of that proposition are equal. Note also that the notion of a type is closely related to that of a set in the sense of Bishop (1967).

The type of natural numbers nat provides a good example of a type. It can be constructed by first establishing that 0 is a canonical term of nat and that $\mathsf{succ}(n)$ is a canonical term of nat for any term n of nat. Then, we say that 0 equals 0 in nat and that $\mathsf{succ}(n)$ equals $\mathsf{succ}(m)$ in nat if n equals m in nat (Martin-Löf, 1984). Those specifications can be symbolically translated as rules of inference:

$$\text{(i)} \quad \frac{}{0 : \mathsf{nat}} \qquad\qquad \text{(ii)} \quad \frac{}{0 \equiv 0 : \mathsf{nat}}$$

$$\frac{n : \mathsf{nat}}{\mathsf{succ}(n) : \mathsf{nat}} \qquad\qquad \frac{m \equiv n : \mathsf{nat}}{\mathsf{succ}(m) \equiv \mathsf{succ}(n) : \mathsf{nat}}$$

Because computation is central to type theory, there must be a distinction between canonical and non-canonical terms in the definition of a type such as nat—otherwise one would have to give an exhaustive account of what counts as a natural number in all cases, thus preventing the possibility of having types at a later stage of construction with elimination rules that return a natural number.

We say that non-canonical terms are terms which do not have an explicit form by which we can directly check that they are the result of the introduction rules of a type, but given a notion of evaluation (a finite reduction of one-step computations), it can be shown that they compute to a canonical term of a certain type. This is one of the key ideas underlying the meaning explanations (Martin-Löf, 1982), an informal semantics for type theory that may be briefly presented as follows:

(i) To know a type is to know that it evaluates to a canonical type;
To know a canonical type is to know

   (a) how to construct a canonical term of that canonical type;

   (b) how to show that two canonical terms of it are equal;

(ii) To know that two types are equal is to know that they have the same terms;

(iii) To know a term of a type is to know that it evaluates to a canonical term of it;

(iv) To know that two terms of a type are equal is to know that they evaluate to equal canonical terms of that type.

The stipulations above are just a small part of the meaning explanations, but they suffice to indicate that type theory has a constructive justification based on the interpretation of terms as constructions, which are in turn viewed as computations. Because it is always possible to extend the present domain of types by stipulating new ways of obtaining terms as possible constructions, the meaning explanations validate Brouwer's conviction that the range constructions is open-ended (Brouwer, 1907). Not surprisingly, the internal representation of Church's Thesis in the meaning explanations is false. Just as it was first envisioned by Brouwer, construction is not an exhaustive notion, for nothing prevents the extension of our underlying computation system with the additional constants and computation rules that follow the introduction of a newly postulated type. There is no restriction to a fixed programming language, thus it is in principle impossible to determine what cannot be computed.

In type theory, membership $a : A$ and member equality $a \equiv b : A$ are treated as primitive forms of judgment. Intensional type theory is so called because the judgmental equality $a \equiv b : A$ has an intrinsic intensional nature. To give an illustration, we consider the addition function, defined by induction on the second argument:

$$\begin{aligned} &m + 0 \equiv m \\ &m + \mathsf{succ}(n) \equiv \mathsf{succ}(m + n) \end{aligned}$$

From the definition of the function, it is immediately clear that $m + 0 \equiv m$ holds but it is not the case that $0 + m \equiv m$ in intensional type theory, for the definition says nothing about it. For the same reason, the equality $m + n \equiv n + m$ does not hold judgmentally.

It is desirable to have a type that acts as an internalization of judgmental equality, since the type theory is unable to express equality propositions otherwise. We deal with propositional equalities via the *identity type*, a type inhabited by witnesses that two terms of a type are the same. Following the pattern above, this type is usually prescribed as follows (Martin-Löf, 1975). For any terms $a, b : A$, the identity type of a and b at A, written $a =_A b$, has reflexivity as a canonical term and the obvious equality relation:

$$\text{(i)} \quad \frac{a \equiv b : A}{\mathsf{refl}_a : a =_A b} \qquad\qquad \text{(ii)} \quad \frac{a \equiv b : A}{\mathsf{refl}_a \equiv \mathsf{refl}_b : a =_A b}$$

But what the elimination rule for the identity type should say? Formally speaking, this question marks the divide between the intensional and extensional type theory. The dichotomy between the two flavors of type theory can be explained in terms of the failure of the reflection of propositional equality in judgmental equality. The intensional elimination rule is designed to keep propositional and judgmental equalities apart without reflection, while the extensional elimination rule treats them as respectively internal and external representations of the same equality relation.

Let us first consider the extensional elimination rule. In extensional type theory, every elimination rule follows the usual pattern that guarantees that all data used in the introduction can be recovered correctly in the elimination. This is known as the inversion principle (Gentzen and Szabo, 1969; Prawitz, 1965). It is described by Gentzen as follows:

> The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only 'in the sense afforded it by the introduction of that symbol'. (Gentzen and Szabo, 1969, p.80-1)

It should be noted that, in constructive semantics, the inversion principle is an implicit assumption underlying the idea that a proposition is specified by stating how to obtain a construction of it (otherwise we would also need to tell how to eliminate them).

Justified by the inversion principle, the elimination rule that is in harmony with the introduction rule for the identity type should state that, given two terms $a, b : A$ and an equality $p : a =_A b$, it must be the case that a and b are judgmentally equal.

$$\frac{a, b : A \quad p : a =_A b}{a \equiv b : A}. \tag{R}$$

This "reflection" rule conflates judgmental and propositional equalities, which makes the former undecidable. Given the above introduction rule for the identity type, this is the only way of conforming to the inversion principle and meaning explanations.[8]

To keep propositional and judgmental equalities apart, the intensional elimination rule has to be weaker than that. Given two terms $a, b : A$, an equality $p : a =_A b$ and a type family P, it states that it is enough to have a term $u : P(a, a, \mathsf{refl}_x)$ in order to define a term of the type $P(a, b, p)$ (Martin-Löf, 1975, §2.7),

$$\frac{a, b : A \quad p : a =_A b \quad u : P(a, a, \mathsf{refl}_a)}{\mathsf{J}_P(a, b, p, u) : P(a, b, p)}.$$

It is not obvious why this should be the correct elimination rule for the identity type. The inversion principle, which also underlies the meaning explanations, tells that the meaning of a type is determined by its canonical terms, that is, it ought to be the case that the terms of a type are (equal to) those introduced by the introduction rules. But when the identity type is considered with the above introduction and elimination rules, the inversion principle becomes false in intensional type theory because the corresponding inference commonly referred to as "K" (Streicher, 1993)

$$\frac{a, b : A \quad a \equiv b : A \quad p : a =_A b}{p =_{a =_A b} \mathsf{refl}_a \ \mathsf{true}}, \tag{K}$$

[8] The reflection rule is sometimes viewed negatively because it is responsible for "loss of knowledge". That is, one might argue that when moving from the premise to the conclusion of the rule the proof-term p gets lost in a way that can no longer be recovered, thus going against the general requirement that in the rules for a constructive theory knowledge should never get lost (Sambin and Valentini, 1998). This is an interesting point, but I believe it incorrectly assumes that terms are always intended to represent proofs. In extensional type theory, terms codify realizers and we should not expect them to carry proof-theoretic information.

does not hold (Hofmann and Streicher, 1998), where A true is a derived form of judgment that can be proved whenever there exists a term a such that $a : A$ is the case. This very fact inspired the development of homotopy type theory, which we discuss in Section 6.

This should not be taken as suggesting that any form of intensional type theory does not adhere to the inversion principle. Without getting into too much trouble, it is possible to change the introduction rule slightly and provide an intensional eliminator that is able to prove (K) (Barzan, 2016). But, like in most variants of intensional type theory, the resulting identity type is still unable to prove fundamental equalities that are true in the meaning explanations. Some of those equalities are crucial for the development of actual mathematics, such as the identification of pointwise equal functions:

$$((\forall x : A) f(x) =_B g(x)) \to f =_{A \to B} g.$$

This function extensionality principle is not provable in most forms of intensional type theory.[9] In extensional type theory (which is so called because, among other things, it proves function extensionality), this is not a problematic principle because the reflection rule allows for a full extensional treatment of propositional equality. Indeed, several equalities which are not provable in the intensional version of the framework are provable in the extensional counterpart.

The normalization theorem, which says that that every well-typed term reduces to a normal form, is usually taken as a justification for intensional type theory, since it implies that the empty type, that is, the type defined by no constructors, does not have an inhabitant, thereby establishing the consistency of the formalism. Normalization provides strong evidence that intensional and not extensional type theory is correct from this perspective—but the point is that this perspective is entirely proof-theoretic. If we wish to take the view of terms as programs seriously then we should not expect that all programs terminate. Especially because programs (constructions) are products of an untyped notion of computation. Now consider the following judgment, where $\Omega := (\lambda x.xx)(\lambda x.xx)$ is an infamous non-terminating term of the lambda-calculus:

$$p : 0 =_{\mathbb{N}} 1 \vdash \Omega : \mathsf{nat}$$

From the standpoint of the meaning explanations, this is a valid judgment because there are no canonical terms of type $0 =_{\mathbb{N}} 1$ (Martin-Löf, 1982). Unlike the intensional counterpart, extensional type theory is designed to reflect this computational aspect of terms in the formalism (in fact the above judgment is provable by the reflection rule). Every formal proof can be expressed as a program, but the converse is not true.

4.2 Cognitive significance

If one wishes to examine cognitive significance via type theory, then, as long as one is fully committed to the meaning explanations, as we are, it appears that there are benefits in considering extensional type theory, for it adequately expresses the view of terms that

[9]Although there are exceptions. See Section 6.

we want to stress: terms, being constructions, are open-ended computations and should not be limited to derivations in a fixed formal system. Unless explicitly stated otherwise, what follows is specifically concerned with extensional type theory.

In light of the distinction between proofs of judgments and constructions of propositions drawn in Section 3 for constructive semantics, it is useful to distinguish, now in the context of a type theory, proofs of typing judgments from terms of a type. It is crucial to tell terms and proofs apart in extensional type theory, especially when the identity type is involved, for, when we have a proof of the judgment $p : a =_A b$, then, by the reflection rule, the term constructed p is reflexivity. But the proof of this judgment need not be trivial. As judgmental equality is undecidable, judgments such as $\mathsf{refl}_a : a =_A b$ do not always have straightforward proofs, since that presupposes a demonstration that both sides of the equality sign are judgmentally equal. In many cases, we are dealing with non-trivial proofs that may appeal to the reflection rule.

An example of a propositional equality that does not have such a straightforward proof may be welcome at this point. Given terms $x, y : A$, a function $f : A \to B$ and an equality $p : x =_A y$ (call this context Γ), in order to construct an inhabitant of $f(x) =_B f(y)$, that is, a reflexivity term $\mathsf{refl}_{f(x)}$, we must first show that $f(x) \equiv f(y)$, but in order to convince ourselves that this judgmental equality is the case, a little ingenuity is required to come up with an argument that serves as an explicit evidence.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash p : x =_A y}\ \text{(CTX)}}{\Gamma \vdash x \equiv y : A}\ \text{(=-ELIM)} \qquad \dfrac{\dfrac{}{\Gamma \vdash f(x) : B}\ \text{(CTX)}}{\Gamma \vdash f(x) \equiv f(x) : B}\ \text{($\equiv$-REFL)}}{\Gamma \vdash f(x) \equiv f(y) : B}\ \text{($\equiv$-SUBST)}}{\Gamma \vdash \mathsf{refl}_{f(x)} : f(x) =_B f(y)}\ \text{(=-INTRO)} \tag{1}$$

In other words, there cannot be a direct reflexivity proof of this equality because there is no immediate proof that $f(x)$ and $f(y)$ are judgmentally equal: any proof will make essential use of the reflection rule.

On the other hand, when compared to the case illustrated above, we can identify a specific class of equality statements that always seem to have simpler proofs—namely, equalities whose proof only depend on the reflexivity, symmetry, and transitivity of equality, renaming of bound variables (α-conversion) or unfolding of definitions. This is shown in the following examples, where the displayed equality proofs appear to be far more mechanical since judgmental equality follows from nothing other than mindless applications of $\equiv$-reflexivity, α-conversion, and definitional abbreviations.

$$\dfrac{\dfrac{}{\Gamma \vdash a \equiv a : A}\,(\equiv\text{-REFL})}{\Gamma \vdash \mathsf{refl}_a : a =_A a}\,(=\text{-INTRO}) \tag{2}$$

$$\dfrac{\dfrac{}{\Gamma \vdash \lambda x.x \equiv \lambda y.y : A \to A}\,(\alpha\text{-EQ})}{\Gamma \vdash \mathsf{refl}_{\lambda x.x} : \lambda x.x =_{A \to A} \lambda y.y}\,(=\text{-INTRO}) \tag{3}$$

$$\dfrac{\dfrac{}{\Gamma \vdash 1 \equiv \mathsf{succ}(0) : \mathsf{nat}}\,(1\text{-DEF})}{\Gamma \vdash \mathsf{refl}_1 : 1 =_{\mathsf{nat}} \mathsf{succ}(0)}\,(=\text{-INTRO}) \tag{4}$$

There is little to no doubt that the equality statements proven above are less informative than the one proven earlier in (1). Indeed, (2) is a trivial consequence of reflexivity, both sides of the equality sign in (3) are identical up to variable renaming, and (4) follows from the unfolding of the definition of 1, which is here taken as constant that abbreviates the successor of zero, $\mathrm{succ}(0)$.

This suggests that the presence or lack of cognitive significance in an equality statement depends on the sophistication of thinking needed in a proof that the equality is true. This can be made more precise as follows:

Definition 4.1 (Cognitive significance)**.** *For any $a, b : A$, the identity type $a =_A b$ has cognitive significance if it does not have a trivial proof.*

An identity type has a trivial proof when it is provable by $=$-introduction, $\equiv$-reflexivity, $\equiv$-symmetry, $\equiv$-transitivity, α-conversion, and definitional rules. Why shouldn't we count other rules such as β- and η-reduction as trivial proofs as well? Calculation involves thinking, no matter how elaborate or subtle. For that reason, computation rules certainly cannot be regarded as obvious modes of proof.

One may object that this theoretical notion of cognitive significance does not properly capture the intuitive notion of cognitive significance. In general, proving a theorem is a task that requires considerable knowledge, focus, ingenuity, time, and mathematical skill. This is no different for equational theorems such as $e^{i\pi} = -1$ which cannot be proven without appealing to advanced mathematical arguments. Why is it that we intuitively recognize that $e^{i\pi} = -1$ carries more information than

$$\int_{\infty}^{0} x^n e^x dx = \int_{\infty}^{0} y^n e^y dy$$

or $\pi = C/d$? Because the latter two equations are trivially true. There is no gain of knowledge because absolutely no mathematical ability is required to demonstrate that those "theorems" are indeed the case.

This notion of cognitive significance gives emphasis to the "mode of proof" rather than the "mode of presentation" of an equality statement. It is always possible to verify that such a cognitively significant equality has difference in sense (see Section 5), but this is merely a consequence of its mode of proof. The notion of proof is, along with that of construction (computation), one of the primitive concepts on which constructive semantics is built. Proofs can be found in mathematics journals or textbooks but, although they can be put into words, proofs are primarily acts for making a judgment known (Martin-Löf, 1985; Sundholm, 1993). More importantly, proofs always go hand in hand with certain constructions, or, better yet, a proof of a judgment A true always exhibits a construction that realizes the proposition A. Brouwer is clear on this point:

> The words of your mathematical demonstration merely accompany a mathematical construction that is effected without words. (Brouwer, 1907, p.127)

Proofs are thus epistemic road maps that guide mathematicians through the conception of a construction that realizes the truth of a proposition. Given their intrinsic epistemic nature, proofs provide a natural solution to Frege's puzzle: the presence or lack of cognitive significance in an equality statement should only depend on what is required to make the equality known. Could there be a more appropriate elucidation of cognitive significance than such an epistemic explanation?

It only remains to be shown that our theory of cognitive significance is adequate. Put another way, how can the knowledge that the type $a =_A b$ is cognitively significant contribute to an extension of knowledge? According to our stipulations, we need to show that an agent who gets to know that $a =_A b$ does not have a trivial proof has a latent ability to make new assertions. For the sake of argument, assume that a construction $u : P(a)$ is known, for a type family P. Then, since the rational subject in question can be justified in knowing that a and b are judgmentally equal, they will be able to infer that $u : P(b)$ and $P(a) =_{\mathcal{U}} P(b)$. Generally speaking, both statements do not have a trivial proof when $a =_A b$ is cognitively significant. Note that $u : P(a)$ and $u : P(b)$ have different assertions because the latter judgment requires a more sophisticated proof by substitution and the knowledge that a equals b.

5 Sense and reference in type theory

There is a suggestion made by Martin-Löf (2001) built on the works of Dummett (1978) and Moschovakis (1994) that the sense of an expression is related to its reference as a program is related to the result of its execution. This "program-execution" view of the sense/reference distinction in type theory fits the computational character of the meaning explanations quite well, so it conforms to constructive semantics.

It is worth noting that, when compared to Frege's theory, the program-execution interpretation of sense and reference has at least three distinctive features, which are discussed as follows. First, the notion of evaluation of a term to its canonical form plays a central role as the mediator of the passage from the sense to the reference of an expression. Since (closed) terms are seen as programs in the meaning explanations, the evaluation of a term can only be interpreted as a program execution. Second, when read constructively, Frege's thesis that the reference of a sentence is a truth value implies a means of deciding whether a type is inhabited or not (Frege, 1892, p.63). It must be rejected and replaced with the idea that the reference of a sentence is a canonical proposition (Martin-Löf, 2001). Third, equality of execution results of programs must be given extensionally whereas equality of programs is intrinsically intensional—suppose a programmer is given two programs in a same language that find prime numbers in a given range, except that the former program gets exorbitantly slower than the latter one for large inputs; they both always yield the same output, but the programmer is most likely to say that the latter program is not the same as the former.

How does the passage from an expression to the sense it expresses takes place? Martin-Löf did not address this question in his essay, but we propose the following. Since $M : A$ is a judgment, before it becomes known to a rational subject, the expression 'M' is an ordinary string of symbols that has no meaning for them. Conversely, although any expression M may in principle be known as a term of some type A, this only happens when one gets to know that $M : A$ holds. This indicates that the passage from an expression to its sense should be determined by the assertion of a typing judgment, and, under the program-execution interpretation, it is tantamount to saying that an expression is only entitled as a program if it has been correctly type-checked.

Now, as Martin-Löf suggested, the passage from a term M (sense) to a canonical term M' (reference) is given by evaluation $M \downarrow M'$ (pronounced "M evaluates to M'"). In brief, the route from an expression to its sense and reference is presented in the same way as its transition to a program and the result of its execution. The situation can be pictured as in the diagram below.

$$
\begin{tikzcd}
 & \text{reference/result} \\
\text{expression} \arrow[ur, dotted] \arrow[r, "M:A"'] & \text{sense/program} \arrow[u, "M\downarrow M'"']
\end{tikzcd}
$$

It is not hard to reconcile this computational interpretation of the sense/reference distinction with the theory of cognitive significance proposed in the previous section: computationally, it is natural to identify two programs that are equal up to renaming of bound variables and definition unfolding because they compute in the same way. This is how equality of sense should be understood. Equality of reference is simply given by propositional equality. So, following Frege, we also have that $a =_A b$ has cognitive significance iff a and b have a different sense but the same reference. The difference is that, as our account of cognitive significance is grounded in the notions of computation and proof, we possess a coherent notion of equality of sense.

Is Frege's equipollence principle still valid under this interpretation? Not exactly, because no internal representation of the principle is possible in a formal system without modalities such as intensional/extensional type theory. Recall that, according to Frege's suggestion, two expressions have the same sense iff they can be substituted for each other in all contexts without altering the truth-value. Type-theoretically, the principle for sentences seems to express the requirement that the types A and B have the same sense just in case we have[10]

$$\frac{a : A}{a : B} \quad \text{and} \quad \frac{a : B}{a : A}. \tag{S}$$

The type-theoretic counterpart of the equipollence principle for singular terms can be naturally expressed by the principle that two (closed) terms $a : A$ and $b : A$ have the same sense just in case, given $C : A \to \mathcal{U}$,

$$\frac{c : C(a)}{c : C(b)} \quad \text{and} \quad \frac{c : C(b)}{c : C(a)}. \tag{T}$$

However, as we saw in Section 4, the criteria (S) prescribes exactly what it means for two types to be judgmentally equal, whereas (T) is a trivial consequence of two terms being judgmentally equal: interchangeability in all contexts. What is going wrong? Those principles obviously go against the identity criterion of senses proposed above. However, a second look reveals that substitution in type theory only occurs in ordinary contexts, unless the framework has means of expressing oblique contexts with the support of modal operators. Without the presence of propositional attitude reports, if a equals b, then there is no doubt that $C(a)$ implies $C(b)$ and vice-versa, regardless of how different the senses of a and b might be.

6 Alternative accounts of cognitive significance

It would be inappropriate to conclude the paper without mentioning the folkloric view that equality of sense and reference coincides, in intensional type theory, with judgmental and propositional equality, respectively.[11] It is not hard to see that this thesis is problematic for at least two reasons.

First, judgmental equality is too coarse to account for equality of sense. To be more specific, judgmental equality is completely oblivious to computational complexity. Every closed term is judgmentally equal to the value they evaluate to. For instance, the recursive definition of

[10] After completion of the present paper the author learned that Sundholm (1994) has proposed a similar type-theoretic interpretation of the equipollence principle for sentences. Sundholm's account is based on Frege's original formulation of the principle, so he did not provide a view for singular terms.

[11] This interpretation is often mistakenly attributed to Martin-Löf. In his actual account, a judgmental equality $a \equiv b : A$ says of the senses of 'a' and 'b' that they are co-referential and a propositional equality $p : a =_A b$ says of the references of 'a' and 'b' that they are equal objects (Martin-Löf, 2001, pp. 14-17). I am grateful to Professor Martin-Löf for clarifying his views on the sense/reference distinction to me.

addition implies that $1 + 1$ and 2 are judgmentally equal, or, to give a more extreme example, the prime factorization $2^4 \cdot 3^2 \cdot 7^2 \cdot 11^3$ is judgmentally equal to 9391536 even though, when run, it becomes clear from the execution time that they do not determine the same program. Intuitively, one wants equality of programs (senses) to be a very intensional notion that only holds for codes with the same computational content—and surely Frege would agree that terms with different computational complexity cannot express the same sense.[12] In contrast, for equality of values (references) the computational behavior of the programs is irrelevant as long as they are observationally equal.

Second, propositional equality is too fine to equate references as far as intensional type theory is concerned. We may require that two functions have the same reference in mathematics when they coincide on all the values of the domain, but, as discussed in Subsection 4.1, in general function extensionality does not hold intensionally.

If one still wishes to take the judgmental/propositional interpretation of sense and reference seriously despite those problems, it appears that one is forced to endow the identity type of intensional type theory with a more extensional structure, as it is done in homotopy type theory (UFP, 2013). The idea is that a type A can be interpreted as a space, a term $a : A$ as a point in A, an equality $p : a =_A b$ as a path from a to b in A, a two-dimensional equality $\alpha : p =_{a =_A b} q$ as a homotopy between paths p and q etc. Because the inversion principle fails for propositional equality in intensional type theory, the identity type can be augmented with new canonical terms without loss of consistency. This key observation opened the door for Voevodsky's univalence axiom, an axiom which implies that equivalent ($\simeq$) types are propositionally equal

$$\mathsf{ua} : A \simeq B \to A =_{\mathcal{U}} B$$

where this notion of equivalence stands for a generalization of the categorical equivalence of ∞-groupoids or the homotopy equivalence of spaces which can be made precise in the language of type theory (UFP, 2013, §§4.2–4.5). As a matter of fact, the full univalence axiom requires that ua be part of the equivalence "equality is equivalent to equivalence".

With the univalence axiom, the resulting type theory is able to refute (K), so it can show the existence of a "non-constant loop", that is, a path $p : a =_A a$ that is not propositionally equal to the constant path refl_a (UFP, 2013, Thm. 3.1.9). One might then propose that a type $a =_A b$ has cognitive significance just in case it is not only inhabited by constant loops:

Definition 6.1 (Cognitive significance)**.** *For any $a, b : A$, we say that the type $a =_A b$ has cognitive significance iff, if a is judgmentally equal to b, there exists an equality $p : a =_A b$ that is not propositionally equal to refl_a, that is,*

$$\frac{a \equiv b : A}{(\exists p : a = b)\mathsf{refl}_a \neq p \ \mathsf{true}}.$$

[12] See Frege (1891, p.29).

(Readers who are familiar with homotopy type theory will immediately notice that this definition says that every cognitively significant type is not contractible with reflexivity as the center of contraction.)

However, even though it can be shown to be adequate, this notion of cognitive significance, which has been suggested in Bentzen (2018), presupposes that propositional equality in homotopy type theory indeed reflects ordinary mathematical equality. While it is open to question whether every mathematician would say that two objects are identical when they are isomorphic, isomorphic objects of a certain field are always indistinguishable within that field, as they share all the properties that matter. Given that point, one might attempt to justify univalence following Awodey (2014) in arguing that it captures a common principle of reasoning embodied in everyday mathematical practice. But this argument is only able to justify the existence of the function ua. Why should one assume that considering equality as equivalent to equivalence (which is what the full univalence axiom states) is a common practice among working mathematicians?[13] Even worse, higher inductive types (generalizations of inductive types that allow for the generation of paths) present an insuperable obstacle to this approach. To see why they pose a problem, consider the following higher inductive type:

$$\text{(i)} \quad \frac{}{\mathsf{base} : \mathsf{S}^1} \qquad\qquad \text{(ii)} \quad \frac{}{\mathsf{loop} : \mathsf{base} =_{\mathsf{S}^1} \mathsf{base}}$$

$$\frac{}{\mathsf{base} \equiv \mathsf{base} : \mathsf{S}^1} \qquad\qquad \frac{}{\mathsf{loop} \equiv \mathsf{loop} : \mathsf{base} =_{\mathsf{S}^1} \mathsf{base}}$$

Explicitly motivated by the homotopical interpretation, the circle type S^1 can be seen as a type-theoretic representation of the unit circle. Nevertheless, it is patently clear that its existence as a type cannot be justified by mathematical practices involving the treatment of the unit circle in terms of equality conditions.

Another problem is that the implementation of univalence and higher inductive types as axioms in homotopy type theory blocks computation due to the fact that they introduce new canonical terms without specifying how to compute with them. This breaks the process of evaluation of a term to its canonical form and, from the perspective of the program-execution interpretation, interrupts the way an expression denotes its reference. Cubical type theory (Cohen et al., 2016; Angiuli et al., 2019), which has univalence, higher inductive types, and function extensionality as theorems, solves this problem but, as the formalism is explicitly based on a sophisticated mathematical structure of cubical sets, it is unclear whether or not it can be given an informal (pre-mathematical) justification. A more serious objection is that homotopy/cubical type theory appears to go against the program-execution interpretation of sense and reference proposed in the previous section: if propositional equality determines equality of reference, then the reference of a term cannot be the result of its execution, for, in general, if $a =_A b$ holds in an empty context in those type theories, then a and b need not evaluate to the same value. This objection applies to any alternative account of how to compute with univalence (outside of the cubical case) that may be found in the future.

[13] See Ladyman and Presnell (2019) and Bentzen (2020) for a more detailed discussion.

7 Implications and further considerations

Before we close this paper, we briefly discuss some features of the theory of cognitive significance we presented, directions in which it could be extended, related work and objections.

It is reasonable to argue that false equality statements may contribute to one's advance of knowledge as well, for the knowledge that a contradictory statement is true certainly allows one to draw any other inferences by the *ex falso quodlibet*, a rule which is usually accepted as constructively valid. As we have seen in Section 2, Frege only ever speaks of true cognitively significant equalities, but because no false equality has a proof in extensional type theory (hopefully), every false equality has cognitive significance in our account. The presence of informative false equalities suggests that we should regard cognitive significance as a property of propositions $a = b$ and not judgments $a = b$ true, for when a judgment that says of an equality that it is true is asserted the equality cannot be false.

One issue worth exploring is whether two logically equivalent propositions should express the same sense or not. Consider the following pair of conjunctions

$$p \wedge q \quad \text{and} \quad q \wedge p.$$

Should they have the same sense? In type theory, this question is not well-posed unless one assumes a stronger form of univalence, "propositional extensionality":

$$(A \to B) \to (B \to A) \to A =_{\mathcal{U}} B$$

that is, the types A and B are propositionally equal if there is a function in the forward direction from A to B and another function in the opposite direction from B to A.

In homotopy type theory, univalence implies propositional extensionality but only for a specific class of types called "mere propositions" (UFP, 2013, §3.3). In intensional and extensional type theory, propositional extensionality is not provable and, as a result, $p \wedge q$ and $q \wedge p$ cannot express the same sense in our account. Does this agree with the intuitive notion of cognitive significance? If we were to agree that logically equivalent propositions had the same sense, then it should be immediately obvious that

$$p \wedge q \wedge p \wedge q \wedge q = p \wedge q \wedge p \wedge p \wedge q$$

and that both sides have the same mode of presentation, but this hardly the case. Perhaps the lesson to be learned here is that it is impractical to find a clear divide between seemingly obvious and challenging logical equivalences.

Rodin (2017) offers an alternative solution for the problem of cognitive significance (in the context of empirical sciences) using the distinction between judgmental and propositional equality in homotopy type theory. Rodin observed that this account is capable of making sense of (i) equality statements $a = b$ where the terms a and b have the same sense expressed by different symbols and (ii) how empirical or other sort of evidences justify the truth of an equality statement $a = b$ when the terms a and b have different senses. (i) is true under Rodin's account if judgmental equality determines equality of sense, given that e.g. $3+1 \equiv 2+2 : \mathsf{nat}$ but both sides of the equality are different as expressions. (ii) also follows from his account, because, in homotopy type theory, the terms of the identity type can be seen as witnesses of identifications between terms. Nevertheless, as it is based on homotopy type theory, Rodin's approach suffers from the basic problems mentioned in Section 6.

Finally, one may object that our account conflates syntax with semantics because it makes cognitive significance depend on the proofs of an equality statement while Frege was primarily concerned with questions of meaning. This objection, however, arises from the failure to notice that Frege's puzzle is about explaining what a cognitively significant equality statement is. We favored an explanation of cognitive significance as difference in mode of proof, but this implies difference in mode of presentation: if sophisticated arguments may be needed to prove that the equality is true, then, because no straightforward equality proof is possible, a and b will have distinct computational content (sense) but the same observable behavior (reference).

8 Conclusion

We hope to have given sufficient argument that our theory of cognitive significance adequately captures the intuitive notion with respect to the presence of equality proofs contributing to one's knowledge. We have also put forward the thesis that extensional type theory provides a close formal representation of the view that constructions are programs, which we take to be one of the basic tenets of constructive semantics. Another contribution of this paper is that it identifies at least two ways in which an equality statement may be said to have cognitive significance: one concerns the mode of presentation of the equality, the other its mode of proof.

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