**The Symbolic Approach to the Omega Rule**

**Abstract.** According to the ω-rule, it is valid to infer that all natural numbers possess some property, if 0 possesses it, 1 possesses it, 2 possesses it, and so on. The ω-rule is important because its inclusion in certain arithmetical theories results in true arithmetic. It is controversial because it seems impossible for finite human beings to follow, given that it seems to require accepting infinitely many premises. Inspired by a remark of Wittgenstein’s, I argue that the mystery of how we follow the ω-rule subsides once we treat the rule as helping to give meaning to the symbol, “…”.

**1. Introduction**

According to the natural deduction form of the ω-rule (Warren 2020),

$$\frac{φ\left(0\right),φ\left(1\right),φ\left(2\right),...}{∀x(φ(x))}$$

In other words, it is valid to infer that all natural numbers possess some property, if 0 possesses it, 1 possesses it, 2 possesses it, and so on. (I assume throughout that the domain of discourse includes all and only the natural numbers, and I must set aside any problems justifying this assumption.) The ω-rule is important because its inclusion in certain arithmetical theories, such as Peano arithmetic and Robinson arithmetic, results in true arithmetic. Such theories are immune to Gödel’s incompleteness theorem, for, in them, every arithmetic sentence is decidable. But the rule is controversial because it seems impossible for finite human beings to follow, given that it seems to require accepting infinitely many premises. Inspired by a remark of Wittgenstein’s, I present the germ of an idea, and my goal in this paper is not to convince you that it is right, but only that it might be worth pursuing. The idea is that the mystery of how we follow the ω-rule subsides once we treat the rule as helping to give meaning to the symbol, “…”. I call this the symbolic approach to the ω-rule, because it treats “…” as a symbol with its own rule-determined meaning. In Section 2, I present the basic idea behind the symbolic approach to the ω-rule. In Section 3, I discuss the metasemantics behind it. In Section 4, I discuss Warren’s (2021) recent defense of the ω-rule. In Section 5, I discuss two objections to the symbolic approach.

**2. The Symbolic Approach: The Basic Idea**

According to the symbolic approach to the ω-rule, “…” is a symbol that the ω-rule helps implicitly to define. This approach is inspired by several related passages of Wittgenstein’s. Note that in the omega rule, infinite premises are *not* written (obviously), nor is a single infinite conjunction of premises (obviously), but several premises followed by the *symbol* “…”, which we often express in English as “and so on”. About such symbols and such natural language expressions, Wittgenstein writes,

The expression “and so on” is nothing but the *expression* “*and so on*” (nothing, that is, but a sign in a calculus which can't do more than have meaning via the rules that hold of it; which can't say more than it shows). That is, the expression “and so on” does not harbour a secret power by which the series is continued without being continued. [...] For the sign “and so on”, or some sign corresponding to it, is essential if we are to indicate endlessness – through the rules, of course, that govern such a sign. […] we calculate with the sign “1, 1 + 1, 1 + 1 + 1 ...” just as with the numerals, but in accordance with different rules. (1974, 282-3, 285, original emphasis)

In a formal system, each symbol must be defined carefully and precisely, yet philosophers, logicians, and mathematicians have not considered the possibility that “…” is itself a symbol in a formal system which must have a rule-determined meaning. They treat it simply as a shorthand, an abbreviation for an infinitely repeating pattern, an infinite repetition of whichever pattern comes before “…”. Elsewhere, Wittgenstein writes that the design of the sign “…” itself tricks us into thinking that it is not a symbol like any other, saying that this would be “less misleading” if we used another sign, such as “Δ” (1976, 170). So, let us here consider the possibility that “…” is a symbol whose meaning is in part determined by the ω-rule. The ω-rule is often considered a version of universal introduction. It is that – and its inclusion in a formal language is partly meaning-constitutive of the universal quantifier, just as the normal introduction and elimination rules are – but it is more importantly a rule of ellipsis elimination.

**3. The Symbolic Approach: The Metasemantics**

 I assume inferentialism about logic and mathematics for the purposes of this paper, which remains the most popular metasemantics of logic, regardless of concerns about categoricity, which I must set aside here (see, e.g., Raatikainen 2008, Murzi and Hjortland 2009). (I would also like to set aside, as much as I can, thorny debates about rule-following. I think it is permissible for me to assume for the purposes of this paper that we do follow rules and that there is some unmysterious explanation of this ability. I don’t think what I argue for here commits me to any particular explanation of that ability.) According to logical inferentialism, the meanings of logical terms are fully determined by the inferential rules governing them (see, e.g., Peregrin 2014). So, e.g., what makes “&” mean *and* (i.e., conjunction) is that it is governed by the introduction and elimination rules for conjunction:

X&Y X&Y X Y

X Y X&Y

Any symbol governed by these rules means *and*, and it means *and* in virtue of being so governed.

According to the symbolic approach, it is surprisingly easy to follow the ω-rule, for there is, on this approach, no important metasemantic difference between the ω-rule construed as ellipsis elimination and, say, conjunction elimination. Just as conjunction elimination helps to define conjunction, the ω-rule helps to define “…”. When “…” is part of the language of a formal system, it must be governed by strict rules like everything else in the language, regardless of whether those rules are made explicit. To channel Wittgenstein, we infer with the sign “*φ*(0), *φ*(1), *φ*(2), and so on” just as with its components, but in accordance with different rules. The ω-rule is one such rule.

Someone who doesn’t use “&” in accordance with the rules for conjunction simply doesn’t mean conjunction by “&”. Similarly, someone who doesn’t follow the ω-rule would mean something else by “and so on” (or by “…”). And it should be obvious that we in fact use the phrase “and so on” in accordance with the ω-rule. (As noted, if the ω-rule is also partly constitutive of the meaning of the universal quantifier, they would mean something else by “all” too, if they didn’t follow the ω-rule.)

**4. The Symbolic Approach Compared to Others**

As far as I can tell, the symbolic approach is different from every other approach to the ω-rule. As noted, most simply deny that we could follow the rule and leave it at that. Others agree but defend the ω-rule as an idealization of our abilities (e.g., Chalmers 2012). Warren (2020, 2021) has recently defended the claim that we actually use the ω-rule. He appeals to a dispositionalist account of belief[[1]](#footnote-1) to claim that we can (dispositionally) believe the infinitely many premises of the ω-rule, and, on the basis of these (dispositional) beliefs, infer the conclusion. According to Warren’s functionalist account of inference, someone’s causal state transition between belief in the ω premises to belief in the ω conclusion is an inference just in case is plays the inference role. A causal state transition plays the inference role just in case the reasoner possesses certain properties: the reasoner must be disposed to cite the content of her premise beliefs when questioned about her conclusion belief, she must not be disposed to judge inferences from the premises to the conclusion to be faulty, and she must be able to exercise some control over the transition (Warren 2022).

Let me make a couple comments about this. First, it is a virtue of my account that it is not committed to any particular account of inference or of belief. Second, I think Warren’s functionalist account of inference is better suited to the symbolic approach than to his own infinite-beliefs approach. For, I think the symbolic approach offers a more intuitive picture of how an ω-inference plays the inference role than his approach. For example, what is really required by the condition that a reasoner be disposed to cite the content of her ω-premise beliefs when questioned about her ω conclusion belief? I am not objecting to the idea of a disposition to cite infinite premises – Warren responds to that objection, and I will not object to his response. I am objecting to the idea that a disposition to cite the premises of (an instance of) the ω-rule *requires* a disposition to cite infinitely many premises. Instead, according to the symbolic approach – and this I think is more intuitive – it is sufficient that the reasoner to be disposed to cite the fact that *φ*(0), *φ*(1), *φ*(2), and so on, where “and so on” is the crucial bit. If I ask someone why she believes that every natural number is *φ*, it is sufficient, to satisfy this condition on inference, for her to say, “because *φ*(0), *φ*(1), *φ*(2), and so on”. She grasps the meaning of “and so on” and makes use of the concept in her justification. That is all it takes to obey the ω-rule. Third, it seems a consequence of Warren’s dispositionalist approach that one can never consciously follow the ω-rule, for the beliefs required to perform an ω-inference are by necessity implicit. I think that is a con of Warren’s approach. Fourth, and relatedly, it seems odd that one can unconsciously follow a rule that one can’t consciously follow. None of these are consequences of the symbolic approach.

I don’t take any of these points, individually or jointly, to be knockdown arguments against Warren. I do think they are disadvantages of his approach from which the symbolic approach doesn’t suffer, and I think the symbolic approach may be more consistent with his own unrestricted inferentialism.

**5. Objections**

Before concluding, I discuss two objections to the symbolic approach. The first objection is that by treating “…” as a symbol governed by its own rules, we have lost the infinitary character of the ω-rule. For, according to the symbolic approach, the premise of (any instance of) the ω-rule is finite! Now, earlier I said that philosophers, logicians, and mathematicians treat “…” simply as a shorthand, an abbreviation for an infinitely repeating pattern. But an abbreviation has a *meaning*. If that’s what “…” *means*, how does it acquire that meaning? Nothing I have said here denies that “…” means an infinitely repeating pattern. I have only argued that, for it to mean that, it must be *by being governed by certain* *rules*, one of which, I submit, is the ω-rule. As Wittgenstein said in the above quote, “…” *does* “indicate endlessness – through the rules, of course, that govern such a sign” (Ibid.). There is nothing strange about this: this is no stranger than the fact that the expression “infinite series” is syntactically finite but means something infinite in virtue of the rules by which it is governed. A belief that has to do with an infinite series – e.g., the belief that series *s* is an infinite series – is not itself infinite. Similarly, an inference from, say, “series *s* is an infinite series” to “series *s* includes more than 3 expressions” does not require forming infinitely many beliefs, although its premise has to do with something infinite. So, there is *something* about the objection that is right: an ω-premise, to which a reasoner appeals in the justification of her belief an ω-conclusion, is finite; but its meaning – what is said by it – has to do with the infinite repetition of a pattern. And this is all that matters. Of course, in a full account more must be said about what it means to mean an infinite pattern – what exactly it is that is meant or is said. The symbolic approach is orthogonal to debates regarding, for example, whether there are any actual infinities or whether all infinites are potential. Wittgenstein, averse to actual infinity, would say that to mean something that has to do with the infinite repetition of a pattern is to mean something that has to do with the endless application of a technique. I think that’s right, and I think that’s the natural view for the defender of the symbolic approach to take, but we needn’t assume that here. We are solely concerned with whether obeying the ω-rule requires forming infinitely many beliefs. According to the symbolic approach, it does not.

A Warrenian may respond as follows: belief in an ω-premise is accompanied by infinitely many dispositions, e.g., dispositions to respond affirmatively to “Is it the case that *φ*(0)?”, “Is it the case that *φ*(1)?”, and so on. This shows that to believe an ω-premise *is* to have infinitely many beliefs. There is a subtle fallacy here. That the belief that *p* is accompanied by infinitely many other beliefs, or even constitutively requires having infinitely many other beliefs, does not show that the belief that *p* *is really* not a single belief but infinitely many beliefs. There are countless examples of this. The belief that series *s* is an infinite series is a single belief. Its possession is accompanied by, and plausibly even constitutively requires, the possession of infinitely many other beliefs and dispositions. Possessing this belief disposes one to respond affirmatively to “Are there more than 0 expressions in *s*?”, “Are there more than 1 expressions in *s*?”, and so on. Counterexamples needn’t be mathematical. Possessing the belief that Macky is a dog plausibly constitutively requires knowing that dogs are not abstract objects and, so, disposes one to respond negatively to “Is it the case that Macky = 0?”, “Is it the case that Macky = 1?”, and so on. So, it is no problem for the symbolic approach that belief in an ω-premise is accompanied by, or even constitutively requires, having infinitely many other beliefs and dispositions.

Another objection is that there is far more to the concept of an infinitely repeating pattern than what the ω-rule determines. That is true. I am not claiming that the ω-rule alone fully determines the meaning of the ellipsis or the phrase “and so on”. I am only claiming that it helps to determine their meaning. It will be fruitful for future work to explore other introduction and elimination rules by which “…” comes to mean an infinite repetition of whichever pattern comes before.

**6. Conclusion**

I have argued that the ω-rule partly determines the meaning of the symbol “…”. This symbolic approach has some advantages over Warren’s recent defense of the ω-rule. It does not defeat the infinitary character of the rule in any objectionable way. In fact, it helps us better understand its infinitary character. There is much work to be done. My intuition is that because the symbolic approach is merely a philosophical account of the ω-rule, it will not affect the *use* of the rule, or the *results* of its use, in any way. For example, the categoricity of Peano arithmetic plausibly still follows with the symbolic approach to the ω-rule. If I am right, true arithmetic is much more easily within our reach than everyone has thought. My hope is that those with more mathematical-logical acumen than me will explore the technical implications of the claims here. To quote David Wiggins out of context, “To this important task I incite those better qualified than I am to undertake it” (1967, viii).

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1. Actually, it is a dispositionalist account of acceptance, which includes belief. I will talk only of belief. [↑](#footnote-ref-1)