SETS AND CLASSES IN PLURAL LOGIC

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Abstract. What is a set? The conventional answer is that it is an extra individual over and above its members. We explore the idea that a set just *is* its many members, but spoken of *as though* they were one thing. The language is full of such pseudo-singular idioms, grammatically singular but semantically plural. We pick on 'multitude' as our all-purpose pseudo-singular noun. It covers both sets and classes, as multitudes that are or are not members of another multitude. The key to making sense of all this is plural logic, whose use of plural variables capable of taking many things at once as values provides the requisite sense in which a multitude of things can be said to be one. The second part of the paper presents an axiomatic theory that explores the hierarchy of levels of plurality with respect to which multitudes are located, and also accommodates without difficulty the inevitable absence of empty and singleton multitudes. It is topic-neutral, having no existential presuppositions and admitting empty terms, in keeping with the fact that a 'theory of multitudes' is only a notational variant of a system of plural logic.

What is a set? The conventional answer is that it is an extra individual over and above its members. We explore the idea that a set just *is* its many members, but spoken of *as though* they were one thing. The language is full of such pseudo-singular idioms, grammatically singular but semantically plural. We pick on ‘multitude’ as our all-purpose pseudo-singular noun. It covers both sets and classes, as multitudes that are or are not members of another multitude. The key to making sense of all this is plural logic, whose use of plural variables capable of taking many things at once as values provides the requisite sense in which a multitude of things can be said to be one. The second part of the paper presents an axiomatic theory that explores the hierarchy of levels of plurality with respect to which multitudes are located, and also accommodates without difficulty the inevitable absence of empty and singleton multitudes. It is topic-neutral, having no existential presuppositions and admitting empty terms, in keeping with the fact that a ‘theory of multitudes’ is only a notational variant of a system of plural logic.

§1 Plural logic, Topic neutrality, Pseudo-singularity.

Here we introduce the three themes that shape our approach to characterising sets and classes. Since we have already written at length on the first two in our book *Plural Logic* (2016), we offer only the minimum on them needed to make the present article self-contained. The third idea—that a syntactically singular term may nonetheless be semantically plural—is only sketched in the book, and is developed altogether more thoroughly here. We should add that it has been fiercely attacked (by Eric Snyder and Stewart Shapiro, 2021), but in our 2024 we reject their critique entirely.

1.1. Plural logic At the heart of plural logic is the relation of plural denotation (plural reference). It relates the semantically plural terms of the language and the worldly items they stand for. The relation itself is plural in the sense that a given term may denote many items at once, not just one or maybe none. Plural logic thus deals naturally not only with the usual singular suspects—proper names and definite descriptions—but also their plural counterparts. Notably, it can handle plural descriptions like ‘the authors of *Principia*’ or ‘the Brontë sisters’ or ‘the first three prime numbers’. Then there are lists, such as ‘Whitehead and Russell’ and ‘Anne, Charlotte, and Emily Brontë’ and ‘2, 3, and 5’, offering another means of denoting several things at once. Another variety are the functional terms, for instance ‘2+3’ or ‘Whistler’s mother’, that denote the values produced when a function is applied to some choice of arguments. When the function is multivalued these terms are typically plural: thus ‘$\sqrt{-1}$’ denotes the two numbers i and $-i$, and ‘Henry VIII’s wives’ denotes six queens. Functional terms are especially significant for logic because of the potential for iteration inherent in them; think how much of mathematics involves the manipulation of terms constructed on the lines of $f(g(h(a)))$, such as $e^{2\pi i}$ or $\log(\sin^{-1}0)$. As to predicates, plural logic enables one to draw the very important distinction between distributive predicates such as ‘is/are even’ or ‘is/are prime’, for which it is analytic that they are true of several things iff they are true of each of them separately, and collective predicates such as ‘is/are consistent’ or ‘is/are infinite’, for which there is no such link.

Perhaps the most important feature of plural formal logic is its use of plural variables. Just as a plural term may denote several items at once, so a valuation may assign several items at once as the values of a plural variable, opening the door to a proper treatment of plural quantification and plural description. We use bold letters **x**, **y**, **z** to stand for plural variables, keeping italic x, y, z for singular variables of the familiar sort. The beginning of the alphabet is used for schematic letters; so for example a and b will stand for arbitrary terms. Note that there is a choice between reading ‘plural’ strictly so as to exclude the singular, and taking it in an inclusive sense by adding in the singular as a limiting case. It is the difference between the strictly plural ‘more than one’ and the inclusive ‘one or more’. The inclusive usage allows orthodox singular variables x, y, z to be sifted out as those plural variables that do not take more than one individual at a time as their value.

Although our book presents a formal system of first-level plural logic, it barely makes a start on higher-level plurality. To do so, one needs to pick out the fundamental logical relation between one item and many, namely *vertical inclusion* or *membership*, expressed by ‘is one of’ and also by ‘is a member of’ where grammar allows it:

Vertical inclusion (*aka membership*)	Emily is one of the Brontë sisters. 2 is one of / a member of the multitude of even numbers. Whitehead is one of / a member of the pair who wrote *Principia*.

See §1.3 for the use of the pseudo-singular ‘the multitude’ and ‘the pair’ here.

Vertical inclusion differs from set membership as standardly conceived in one essential respect, namely the nature of the arguments they take. Both the arguments of orthodox set membership are individual things, one an urelement or a set, the other necessarily a set. By contrast, for vertical inclusion the first argument may be any item, individual or multitude, provided only that the second is of a higher level of plurality. That is why we call it ‘vertical’ inclusion—‘Emily is one of Anne’ or ‘2 is a member of 2’ would be false at best. The two relations are however structurally analogous in several ways (irreflexivity, asymmetry, and non-transitivity in both their singular and plural forms), and one need have no qualms in following Russell’s lead in using $\in$ to symbolize ‘is one of’ (1903, §489).

Vertical inclusion can be used to define two other key relations, both of which are lateral in the sense that they either permit (in the case of lateral inclusion) or require (in the case of identity) the related items to be of the same level of plurality. The first of them is *lateral inclusion* as expressed by ‘are some of’ or maybe ‘are among’:

Lateral inclusion	2, 3, and 5 are some of the prime numbers. Emily and Charlotte are among the Brontë sisters.

It reappears in §6 under the name ‘submultitude’ and symbolized by $\subseteq$. The other lateral relation is plural identity

Plural identity	The authors of *Principia* are Whitehead and Russell.

When ‘plural’ is understood strictly, plural identity $a{=}b$ is definable as $a{\subseteq}b \wedge b{\subseteq}a$. In our own formal treatment (§4), however, we treat it inclusively, and so need to take = as primitive since it now covers singular identity too.

All these relations carry over to higher levels of plurality. So we have

Vertical inclusion	3 and 5 are one of the pairs of twin primes.
Lateral inclusion	3 and 5, 5 and 7, and 11 and 13 are some of the pairs of twin primes.
Plural identity	The authors of multivolume classics on logic are Aristotle, Frege, Whitehead and Russell, and Hilbert and Bernays.

The plural definite descriptions in each example, and the (nested) lists in the last two, are all second-level plural terms.

1.2. Topic neutrality Formal logic has traditionally been conceived as the study of those aspects of arguments that are independent of any particular subject-matter or 'topic neutral', to use the phrase coined by Gilbert Ryle (1954). The current predicate calculus, however, fails to be topic neutral in one vital respect. For built into it is the assumption that the universe of discourse—the domain of individuals—is nonempty. This leads to dubious argumentation. Consider for example the attempt by three leading set theorists of the later 20th century to demonstrate the existence of an empty set. Their idea was to use separation or replacement to derive it from the existence of any set chosen at will. Where, one might ask, does this initial set come from? The answer, as given by Joseph Shoenfield (1977, p. 328) and Robert Vaught (1985, p. 67), is 'one can use the usual axioms of logic to conclude that there is at least one set', or 'by logic there is some set' The third member of the trio, Azriel Levy (1979, p. 20), gives the game away: 'since $\exists x(x{=}x)$ is a theorem of first order logic, and since our only objects here are sets, we can say that we assumed the existence of at least one set when we decided to adopt first order logic here.'

We cannot accept a logical system that invites and legitimises such question-begging. But then one needs to devise an acceptable replacement. From the outset, however, this project has been dogged by a difficulty over free variables (see Jaśkowski 1934). For, given that a valuation assigns a value to every variable, in an empty domain there will be no valuations, making every open formula come out as vacuously true. So if A is an open formula like $x{=}x$, which is true in every non-empty domain, then both A and $A{\rightarrow}\exists xA$ will come out as true in every domain, but $\exists xA$ will not. Some reformers have resorted to expedients as desperate as disallowing modus ponens or invoking a mysterious 'null thing' or 'outer domain'. But in fact all that is needed is to modify the standard idea of a valuation so that variables as well as proper singular terms may either take an individual as value or may receive no value; for the detailed implementation of this solution see §5 below. In our book we used this idea to prove the soundness and completeness of a system we called 'singular logic', with modus ponens as its sole rule of inference and as axioms the following schemes, both as they stand and prefaced by any number of universal quantifications. Note $\mathsf{E}!a$ abbreviates $\exists x(x{=}a)$ and symbolises existence, while ι is the description operator.

(i) A where A is tautologous
(ii) $\forall x(A{\rightarrow}B) \rightarrow (\forall xA \rightarrow \forall xB)$
(iii) $A \rightarrow \forall xA$ where x is not free in A
(iv) $\forall xA(x) \rightarrow (\mathsf{E}!a \rightarrow A(a))$ where $A(a)$ has free a wherever $A(x)$ has free x
(v) $\forall x\, x{=}x$
(vi) $a{=}b \rightarrow (A(a) \leftrightarrow A(b))$ where $A(b)$ has free b at zero or more places where $A(a)$ has free a
(vii) $(\neg\mathsf{E}!a \wedge \neg\, \mathsf{E}!b) \rightarrow (A(a) \leftrightarrow A(b))$ where $A(b)$ has free b at zero or more places where $A(a)$ has free a
(viii) $a{=}b \rightarrow \mathsf{E}!a \wedge \mathsf{E}!b$
(ix) $\forall y(y{=}\iota xA \leftrightarrow \forall x(A \leftrightarrow x{=}y))$ where y does not occur in ιxA

This explain why it is these schemes, with plural variables, that reappear as the 'elementary axioms' of §8.1.

1.3. Pseudo-singularity We start with a puzzle of a type first presented, but batted away, by Russell (1903, §71). It is perfectly correct to say that Whitehead and Russell were the pair who wrote *Principia*, but whereas 'Whitehead and Russell' is, in Russell's words, 'essentially plural', 'the pair' is singular. What is going on?

We all learn about nouns like 'scissors' or 'pants', which (i) only occur in the plural, yet (ii) head terms standing for one thing, like 'the scissors' or 'my pants'. Grammarians pick up on (i) and call them 'pluralia tantum' (plurals only), but we prefer to emphasise (ii) and call them 'pseudo-plurals'. Now we meet almost the opposite phenomenon. 'The pair who wrote *Principia*' may be grammatically singular but, we claim, it is actually plural. It does not stand for an extra individual such as a whole, but stands for the two men themselves, just like the explicitly plural 'The authors of *Principia*' or the list 'Whitehead and Russell'. In short, it is *pseudo-singular*. The same is true of the whole family of nouns of multitude running up from 'pair' through 'trio' and 'foursome' and so on. Our all-purpose choice of pseudo-singular noun will be 'multitude', the least specific and so most versatile of them all and the word we think best translates Cantor's 'Vielheit'; others may say 'plurality' or 'multiplicity'. Since we shall want to speak of multitudes and individuals in the same breath, we need a neutral word that will cover both. The word we choose, as carrying the least baggage with it, is 'item'.

We now need to spell out and defend our claim. Its origins, we suggest, lie in a striking limitation in expressive resources that English suffers, in common with many other languages. We all know how to pluralize a noun—apart from a few irregular formations, one simply adds an 's'. But it is not possible to repeat the procedure in order to further pluralize a noun that is already plural: the language has no 'thingss' or 'thingsss' to follow 'things'. How then to proceed? One's first thought is to replace the target expression by a semantically equivalent but grammatically singular one, which can then be pluralized in the usual way. For example, 'the two men who wrote *Principia'* can be replaced by 'the pair of men who wrote *Principia*', which can then be pluralized as usual to give 'the pairs of men …'. Or again, 'some prime numbers' can be replaced by 'some multitude of prime numbers', which can then be pluralized to give 'some multitudes of prime numbers'. Moreover, once we have hit on the idea we can simplify and extend it by applying it from the start, beginning with the singular '*N*' and pluralizing it by going straight to 'multitude of *N*s' and thence to expressions of ever-higher levels of plurality: pairs of multitudes, multitudes of multitudes of multitudes etc.

We can describe the proposed procedure roughly as the replacement of a plural noun '*N*s' by 'multitude of *N*s'. But it cannot always be a simple substitution, for the obvious reason that the grammatical rules of concord will be a constant obstacle. It is therefore necessary to look beyond the noun itself to the immediate linguistic context in which it occurs. So we start with the various determiners that serve to fix different roles for the accompanying noun. The relevant ones are of two types, according as they can go with both singular and plural nouns (call this type D_1), or only with plurals (D_2). D_1 includes *the*, *some*, *any*, *no*, *my*, *your* etc. D_2 includes *all*, *many*, *several*, *most*, *few*, *both*; *two*, *three* etc. Correspondingly, the problematic noun phrases fall into two classes, of the form (1) D_1 + *N*s and (2) D_2 + *N*s. For (1) one can indeed simply replace '*N*s' by 'multitude of *N*s'. For (2) a preliminary addition of 'of the' will bring them into line with (1), the point being that instead of e.g. 'both *N*s' one can equally well say 'both of the *N*s'.

Once 'multitude of *N*s' is recognized as an iterable alternative to '*N*s', one sees that this also resolves a quite different problem of pluralization. For as well as the two types of determiner described above, there is a third, call it D_3, consisting of those that only go with singular nouns. They include *each*, *every*, *neither*, *a*/*an*, *one*. And phrases of the form D_3 + *N* are problematic because although normally there is of course no difficulty in pluralizing a singular noun, these contexts are exceptional—'each things' is ungrammatical. The solution is to use the alternative procedure in which 'each *N*' is pluralized as 'each multitude of *N*s'.

If we pause at this point, it is not because there is no more to say. We have said nothing about languages other than English, or the history of language. All we have offered so far is a hypothesis, a rational reconstruction of the current situation. What we need is more evidence that it is true. It would be a waste of effort to trawl through all the varieties of noun phrase. We focus on *terms*, noun phrases which denote some item or items that can be identified uniquely in the contextual or general knowledge shared by speaker and hearer, failing which they denote nothing. The wording is taken from Quirk's account of 'definite reference' (1985: p. 265), but we have added a mention of empty terms, since 'the greatest prime number' surely belongs to the same lexical class as 'the least prime number', and we have emphasised the possibility of a term denoting many things at once. Our plan is to throw more light on nouns of multitude by exploring the denotational behaviour of the terms they head.

A term may be classed as singular or plural in more than one way. First, it may be *grammatically* singular or plural. Then it may be *actually* singular or plural on an occasion, according as on that occasion it denotes at most one individual or many items at once (to get a comprehensive dichotomy, 'actually singular' includes empty terms as well as terms denoting a single individual). And it is *semantically* singular or plural according as there can only be occasions on which it is actually singular, or it is possible for there to be an occasion on which it is actually plural. Finally, we say that a noun is semantically singular if every term that it heads is semantically singular; it is semantically plural if it heads at least one semantically plural term.

We start by presenting a test for whether a term *a* is semantically singular or plural. It turns on the validity of this inference scheme, call it the or-rule

From ⌈*F* or *G*⌉*a* infer *Fa* or *Ga*.

The use of symbols here calls for explanation. *Fa* and *Ga* stand for sentences with subject *a* and distributive predicates *F* and *G* respectively. We treat predicates as lexical items, so that e.g. *is even* and *are even* are regarded as different morphological forms of the same predicate. ⌈*F* or *G*⌉*a* stands for a sentence whose predicate is obtained by linking *F* and *G* by *or* (coordination), while the ceiling brackets ⌈ ⌉ indicate that the result has been *reduced* as far as possible by the pruning (ellipsis) of shared material, as may be seen below in 'is odd or ~~is~~ even', 'are odd or ~~are~~ even', 'were educated at boarding school or ~~were educated~~ by private tutors'. If there is no shared material there is no further room for reduction, but that is not an issue here, since for every *a* there is always a potential infinity of *F* and *G* with shared material: just think of all the predicates that begin *is/are*, or the possibility of using the contrary of *F* to play the part of *G*.

There is nothing artificial about reduction—on the contrary, the maxim for good usage is 'reduce where possible' (Quirk 1985: pp. 860, 927). To grasp its importance one need only see what happens if it is ignored. In his book on Frege's philosophy of language, Michael Dummett tries to define what it is to be a Fregean 'proper name' by presenting a series of negative tests, each designed to rule our certain expressions, in the hope that they will do the job when taken together. His test to rule out plurals turns on the validity of the inference from 'It is true of *b*'s either that *A*(they) or that *B*(they)' to 'Either it is true that *A*(*b*'s) or it is true that *B*(*b*'s)' (1973: p. 60). But as we read his premise, with its repetition of *that* and the way it puts *that* within the scope of *either* and *or* rather than vice versa, it amounts to the disjunction of two predications rather than the predication of a single disjunctive predicate. As such, it is equivalent to the conclusion, making the test useless. A proper test would use reduction to achieve the right reading, as in the case were *b*'s is *prime numbers* and *A*( ) and

B( ) are *are odd* and *are even*, when the right premise would be 'It is true of prime numbers that they are either odd or even'.

Turning now to the application of our test, first, whenever *a* is an ordinary grammatically singular term the or-rule is obviously valid, and one illustrative example will suffice. Let *a* be *the least prime number*, let *F* be *is odd* and *G* be *is even*, so that ⌈*F* or *G*⌉ is *is odd or even*. The test inference is thus from 'the least prime number is odd or even' to 'the least prime number is odd or the least prime number is even', and this, as we said, is clearly valid. Next, let *a* be grammatically plural. Here we need to distinguish two cases: (i) *a* is semantically plural, (ii) *a* is semantically singular. In case (i) the or-rule fails at the first hurdle. Thus let *a* be *the prime numbers*, and let *F* be *are odd* and *G* be *are even*, so that ⌈*F* or *G*⌉ is *are odd or even*. Then the inference fails, for it is true that the prime numbers are odd or even, but not that the prime numbers are odd (2 is an exception), nor that the prime numbers are even (2 is the only example). Case (ii) is less common, but still significant. It is where the term is semantically singular for the special reason that it is doomed to be always empty. For examples we turn to nouns with contradictory or self-contradictory modifiers, such as feature in *the odd even numbers* or *the non-existent occupants of the building*. Here the or-rule holds good, but for the special reason that its premise is necessarily false.

Now take our putative cases of pseudo-singularity. First let *a* be *the multitude of prime numbers*, with *F* and *G* as before. It is true that the multitude of prime numbers is odd or even, but not that it is odd, nor that it is even, and the inference fails. Next let *a* be *the pair who wrote Principia*, let *F* be *was educated at boarding school*, and *G* be *was educated by private tutors.* We know that Whitehead went away to Sherborne while Russell was home schooled. So it is true that the pair who wrote *Principia* were educated at boarding school or by private tutors; that is to say, '⌈*F* or *G*⌉*a*' is true. (The switch from *was* to *were* is a case of what grammarians call 'plural override', a characteristic of British English; a speaker of American English might not make the switch.) But neither *Fa* nor *Ga* is true and so '*Fa* or *Ga*' is false. In short, in both cases the inference fails, and moreover *fails for exactly the same reason that it fails for an explicitly plural term*. Lastly, we add the case where *a* is *the multitude of non-existent occupants of the building*. Here *a* is doomed to be empty on every occasion, no matter how many occupants there actually are, and the or-rule is valid because its premise is necessarily false—again, just like the case of the corresponding explicitly plural term.

Both blocs of examples can stand as representative of countless others, and we take them as providing the desired support for our two contentions. First, we take it as now established that the validity or invalidity of the or-rule is a good test of a term's being semantically singular or plural. Second, and using this test, we take it as now established that a grammatically singular noun of multitude such as 'multitude of *N*s' behaves semantically in just the same way as the conventionally generated plural '*N*s', give or take such inevitable discrepancies as the use of the plural idiom in generic propositions.

Our conclusion, then, is that talk about multitudes embodies a systematic mismatch between syntax and semantics. Logicians should not be surprised by this. For what is the modern badge of their profession if not the recognition of a systematic mismatch between the syntax and semantics of 'everything', 'something' and 'nothing', and the construction of a whole new language of quantifiers and variables to put the matter right? Now it is the turn of set theory to undergo the same sort of revision. When faced with a mismatch of this sort, the logician typically has a choice of two strategies. One is to continue with the grammar of their natural language, while using the resources of their logic to illuminate the semantical situation. The other is to create a fresh language (or at least an illustrative fragment) whose syntax has been reworked to match the semantics. We follow the first strategy in §§3 and 4, before developing the second thereafter. We are not the first in the field: pseudo-singularity is there in Russell's 'the class as many' (see §2.1 below). But we have something Russell did not have, namely access to a functioning plural logic.

§2 Multitudes.

2.1. The hierarchy of multitudes The outcome of our discussion of pseudo-singularity is that a multitude *is* its many members, but spoken of *as though* they are one and treated accordingly. As we said earlier, we are not the first to take this approach to set theory. In *The Principles of Mathematics* Russell debates the merits of the idea of a 'class as many' (1903, §70ff and §489ff). By and large this is the same as what we mean by a multitude (though for a more critical look see our 2016, §2.5). In quoting him we have therefore taken the liberty of substituting our terminology of 'multitude' and 'individual' for his 'class' and 'term', as well as using plural variables in formula (2) below to give effect to what would surely have been his intention had he got such a resource to hand. His great moment of clarity comes right at the end of the book:

> Although a [multitude] is many and not one, yet there is identity and diversity among [multitudes] and thus [multitudes] can be counted as though each were a genuine unity; and in this sense we can speak of *one* [multitude] and of the [multitudes] which are members of a [multitude] of [multitudes]. *One* must be held, however, to be somewhat different when asserted of a [multitude] from when it is asserted of an [individual]; that is, there is a meaning of *one* which is applicable in speaking of *one* [*individual*], and another which is applicable in speaking of *one* [*multitude*], but there is also a general meaning applicable to both cases. (1903, §490).

This can do with some unpacking. Russell has just explained (§489) what he means by 'there is identity and diversity among [multitudes]', namely that they are the same or different according as they have the same members or not. And he has previously (§128) defined 'one' as a quantifier, which we symbolize as $\exists_1$. His pattern of definition can be spelt out to give different versions, according as the relevant variables are taken to be (1) singular or (2) strictly or inclusively plural.

(1) $\exists_1 x A(x) =_{df} \exists x A(x) \wedge \forall x \forall y (A(x) \wedge A(y) \rightarrow x{=}y)$

(2) $\exists_1 \mathbf{x} A(\mathbf{x}) =_{df} \exists \mathbf{x} A(\mathbf{x}) \wedge \forall \mathbf{x} \forall \mathbf{y} (A(\mathbf{x}) \wedge A(\mathbf{y}) \rightarrow \mathbf{x}{=}\mathbf{y})$

We say that (1) delivers Russell's 'there is a meaning of *one* which is applicable in speaking of one individual'. We say too that if the variables in (2) are read as strictly plural then it delivers his 'and another which is applicable in speaking of one multitude', while if they are read as inclusively plural it delivers his 'there is also a general meaning applicable to both cases'. We note too that $\exists_1 \mathbf{x}\ \mathbf{x}{=}a$ is true for any multitude a; in other words every multitude is one in the appropriate sense. Crucially, this is enough to justify speaking of 'the multitudes which are the members of a multitude of multitudes'.

It is a great pity that Russell never developed his thoughts on plurals beyond the remarks we have cited. Instead, he promptly repudiated both halves of

> the fundamental doctrine upon which all rests … that the subject of a proposition may be plural, and that such plural subjects are what is meant by classes which have more than one term. (1903, §490)

Such violent about-turns are characteristic of Russell, but this one may have been precipitated by a pair of far-reaching argumentative missteps which led him to reject multivalued functions and accept singletons (see our 2016, §§9.4 & 14.4).

With this introduction, we are in a position to say more about multitudes. We all know that Cantor defines a set as a collection of many things into one whole. More specifically, he says it is a separate thing (Ding für sich), a unified whole of which its members are components or constituent elements (1932, pp. 282 & 379). This, we argue, is where 'Cantor's Paradise' reveals its resident Serpent. Cantor needs a sense in which a set is one but can only find it by bringing in an extra individual, his 'unified whole'. By contrast, it can hardly be said too often that a multitude is *not* an extra individual over and above its members. Though one in the appropriate sense, it is nothing more or less than its many members. The theory of multitudes just is plural logic, treated in a thoroughly pseudo-singular manner. (Max Black may be mentioned here for his Russell-inspired polemic 'The elusiveness of sets', 1971; for our assessment of it, see our 2016, §2.6. We may also mention Peter Simons' 'The ontology and logic of higher-order multitudes' (2016). After a gentle introduction to the topic, he offers a prospectus for a Leśniewski-inspired theory based on a novel membership relation, with the result that individuals go from having no members to having themselves as sole member. See §2.4 of our 2016 for our assessment of his claim that Leśniewski's 'logical system called "ontology" contains a theory of multitudes in the first-order fragment' (p.61).)

The picture that now guides us is of speaking of given individuals one at a time, also of many at once (a multitude), of many such 'manys' (a multitude of multitudes), and so on up into the transfinite. There is no quasi-temporal talk of 'stages' or 'formation' in this picture. As soon as the individuals are given, all the multitudes of individuals are given with them, and the multitudes of multitudes etc, all there too and only awaiting recognition. But before developing this idea we have to allay a fundamental doubt: can we be sure that our conception of multitudes allows for such a hierarchical structure? Oliver Tatton-Brown has helpfully spelled out for us the grounds for doubt in the shape of a hypothetical objection, which we now paraphrase. Consider a supposedly second-level multitude, for example the pair of pairs of twin primes, 3 and 5 and 5 and 7; call it *a*. Then, says the hypothetical objector,

> On your thesis, *a* is its members. But by the same token, its members are its members' members. Doesn't it follow, by the transitivity of identity, that *a* is its members' members, namely the prime numbers 3, 5 and 7—so not a second-level multitude nor even a pair?

Our response is that this contains a subtle fallacy of equivocation. In the opening sentence the phrase 'its members' can be replaced by 'the members of *a*', referring to the many members together. It cannot, however, be replaced by 'each member of *a*', referring to the members taken separately, since the relation of plural identity is emphatically not distributive: Whitehead and Russell are Whitehead and Russell but they are not Whitehead and they are not Russell. Whenever 'its members' occurs subsequently, however, if it is reduce the level of plurality it must be read as equivalent to 'each member', referring to the members taken separately. In short, the repetition of occurrences of the same phrase is deceptive, vitiating any appeal to transitivity. We called the fallacy a subtle one, because the distinction between the two readings of 'its members' is so easily overlooked. In our formal language below, however, it is clear as anything could be. 'The members of *a*' remains a genuine plural term, symbolized $\mathbf{x}\cdot\mathbf{x}\in a$ (see *Exhaustive description* in §6), but 'each member of *a*' becomes an incomplete symbol, a fragment of a quantified expression $\forall\mathbf{x}(\mathbf{x}\in a \rightarrow \ldots)$.

As we said above, the picture that guides us is of speaking of given individuals one at a time, of many of them at once, of many such multitudes, and so on by way of ever more plurally plural layers. We envisage each such layer as superimposed on the accumulation of the lower ones, to form cumulative *levels* (in orthodox set theory, this was Dana Scott's name (1974) for the cumulative segments or *Abschnitte* of Zermelo's 1930). This cumulativity allows for multitudes with members drawn from mixed levels, as with the authors of multivolume classics in §1.1. It can be shown that levels are well-ordered by vertical inclusion (see §10.1), so the effect is to impose a clearly marked vertical dimension on the overall scene.

2.2. Empty and singleton multitudes? On our account a multitude just *is* its members, so a multitude with no members would be nothing, i.e. nonexistent. A multitude with exactly one member *a* would likewise be identical to *a*. But it would follow that $a \in a$, and this is ruled out by the irreflexivity of $\in$. In arguing this way we are not proscribing empty and singleton multitudes, merely drawing attention to their nonexistence.

Is this discovery damaging, should it be troubling? Not at all. We are aware of four places where there has seemed to be a need for an empty set or singletons, and we have criticized various attempts to make sense of them (see our 2018, §2). Our purpose here, however, is not polemical, but only to show how in each case the difficulty can be handled in an alternative way. But it is worth reminding the reader that the issue is only the visible tip of the iceberg. The big difference between orthodox set theory and a theory of multitudes is a conceptual one, which requires a switch of *gestalt* from thinking of a set as another individual to seeing it as the multitude of its members.

Intersection. This is the prime example. The idea that the intersection of several sets should always be the set of their common members has led to positing an empty and singleton sets to be the intersection in the case where there is no common member or only one. But with multitudes there is an equally natural alternative. For in general the multitude of all the common members just *is* those common members, which suggests taking them to be the intersection in every case. So when there are no common members we take the intersection to be nothing, zilch. And when there is exactly one, we take the intersection to be it. To avoid any misunderstanding, we are not positing an empty multitude, nor identifying a singleton multitude with its sole member. It is a matter of judicious replacement rather than (mis)identification. See §6.

Separation. The classical principle of Separation envisages the separation of any number of members from a set, and asserts that the separated members always form a set too. Cantor proposed to restrict this assertion to the case where the number of separated items is more than one, and expressed this in a lapidary manner: 'Every submultitude of a set is a set' (*Jede Teilvielheit einer Menge ist eine Menge*: 1932, p. 444). All we need to do is to follow his lead: see §8.3.

Ordered pairs. Kuratowski's definition is not suitable for us, since it relies on singletons. The definition we propose as an alternative is a variant of the one Hausdorff presents in his 1914, pp. 32–3. That calls for a pair of individuals as markers, which he calls 1 and 2 but whose identity is arbitrary. We propose to use as markers the two lowest levels, V_1 (the individuals) and V_2 (the individuals plus the various multitudes of individuals). See §10.6.

The foundational role of set theory. Many mathematicians are happy to give a foundational role to the natural numbers supplemented by set theory; see for instance Landau's classic treatise *Grundlagen der Analysis* (1930, p. 1). Writers of a strongly reductionist persuasion, however, are attached to the idea of defining the natural numbers as 'pure' sets. Since there is no such thing as a pure multitude we offer an alternative process of reduction, under which the natural numbers, starting with the first, 0, are identified with the successive finite levels, starting with the first, V_1. See §10.5.

The last two of these proposals rely on the same assumption, namely that there are many (at least two) individuals. On the face of it this assumption, to which we give the honorific title of the *Axiom of Plurality*, is a very modest proposition. After all, the classical predicate calculus is already committed to there being at least one individual, and to double that commitment is no more than to endorse a truism (shades of G. E. Moore's 'Here is one hand and here is another'). The impact of the Axiom of Plurality on the theory of multitudes, however, is dramatic. For if it fails to hold, there are no multitudes at all, but whenever it does hold, there are infinitely many. It does not qualify as a logical truth, however, since it violates the principle of topic neutrality which we take to be the criterion of the genuinely logical. So we confine it to its minimum role as a hypothesis for those propositions that depend on it.

§3 Sets and classes.

Some multitudes are *elements*—members of a further multitude—while, it turns out, others are not. We call them *sets* and *classes*, respectively.

3.1. Sets Each set is located within a transfinite, cumulative hierarchy of levels (using 'level' now in a defined sense). A level is itself a particular kind of set. We may index levels with ordinals, taking V_1 to be the first level, and generally V_α, V_β etc, with the hierarchy defined by transfinite recursion. The guiding idea is that a level is the set of all the individuals together with all the members and subsets of all lower levels, where V_α is lower than V_β if $\alpha<\beta$.

Since it has no lower levels, the first level V_1 is simply all the individuals, assuming there are more than one. The second level V_2 is obtained from the first by application of the *power-plus* operation, which maps any multitude to the multitude of all its members and subsets. Every higher level V_α is the union of the power-plus multitudes of all levels lower than V_α. Anticipating the notations for exhaustive description, individual and power-plus to be explained in §6, the hierarchy is defined by

$$V_1 = \mathbf{x}\cdot U\mathbf{x}$$
$$V_2 = P^{+}(V_1)$$
$$V_\alpha = \cup_{\beta<\alpha}(P^{+}(V_\beta)) \qquad \text{where } \alpha \text{ is any ordinal} > 2.$$

An equivalent characterization starts with V_1 as all the individuals, then defines $V_{\alpha+1}$ as the power-plus multitude of V_α, with unions taken at limits

$$V_1 = \mathbf{x}\cdot U\mathbf{x}$$
$$V_{\alpha+1} = P^{+}(V_\alpha) \qquad \text{where } \alpha \text{ is any ordinal}$$
$$V_\lambda = \cup_{\beta<\lambda}V_\beta \qquad \text{where } \lambda \text{ is any limit ordinal.}$$

Each set is a member of some level higher than the first. The choice of individuals, comprising the first level, determines the make-up of every higher level, and so completely determines what sets there are. To illustrate, suppose there are just two individuals a, b. Then the first three levels are as follows (we use $|a, b|$ to denote the multitude whose members are a and b, so that $|a, b, |a, b||$ denotes the multitude whose members are a, b, and $|a, b|$, and so on).

V_1 includes		V_2 includes		V_3 includes	
	a		a		a
	b		b		b
			$\|a, b\|$		$\|a, b\|$
					$\|a, \|a, b\|\|$
					$\|b, \|a, b\|\|$
					$\|a, b, \|a, b\|\|$

Note that although $V_{\alpha+1}$ is $P^+(V_\alpha)$, the number of members of $V_{\alpha+1}$ is not, in general, the number of members of V_α plus the number of subsets of V_α, since this would double count those subsets of V_α which are also its members. Thus in our example V_3 has 6 members, not 7, since $|a, b|$ is both a member and subset of V_2. To obtain the correct rule, we use the equivalent characterization of $V_{\alpha+1}$ as $\mathbf{x}\cdot(U\mathbf{x} \vee \mathbf{x}\subseteq V_\alpha)$, that is to say, the multitude of all the individuals plus all the subsets of V_α (see §11, Theorem 31). Then the rule is that if there are m individuals (for $m > 1$), then V_1 has m members; and if a level V_α has n members, $V_{\alpha+1}$ has $m+2^n-n-1$ (we need to subtract both 1 and n, since there are no empty or singleton multitudes). Thus starting with finitely many individuals, the first limit level V_ω is the first level to have infinitely many members.

If $V_{\alpha+1}$ is the lowest level of which a particular item is a member, we call α the *rank* of the item, which may be regarded as a measure of its height within the hierarchy. It follows that each V_α is the multitude of all items with rank less than α. In particular, V_1 is the multitude of all items with rank 0, i.e. the set of all individuals.

3.2. Classes When and why might something be a class rather than a set? The answer offered in the literature is von Neumann's doctrine of 'limitation of size' (1925), but the idea is now known to go back to Cantor himself. In his 1899 letter to Dedekind (1932, p. 443), he describes a multitude as 'inconsistent' if the idea of an assembly (Zusammensein) of all of its members leads to a contradiction. Though the language is a little different from the 'collection into a whole' of his published statements, the import is the same. Such a multitude cannot be treated as one, and a fortiori is not 'one of' anything. An item fails this test when it is at least as big as the class of all ordinals (Cantor), or when it has as many members as there are items of all sorts (von Neumann).

Both writers thus see the trouble as lying in the excessive size of classes, though neither says why size should matter. A reason for thinking that size does not matter is to be found by exploiting the extra degree of freedom offered by making different choices of individuals; cf. the 'higher set theories' of von Neumann's article. Suppose somebody chooses to represent a state of affairs in which there are such-and-such individuals, and that their notion of unacceptable size (the number of all items, say) is consequently n, say. Now suppose that somebody else chooses to represent a state of affairs in which there are n individuals. If size is what matters, their choice will be ineligible, which to our mind is absurd. To a logician, what matters about a multitude is its height, not its size.

In any case it is completely misguided to look for something about a multitude, like Cantor's 'inconsistent' or von Neumann's 'too big', that makes it unfit to be collected into one thing. As we have seen, *every* multitude is one in the same appropriate sense; there is no further test of 'collectability' for it to pass or fail. No: if a multitude is not an element, it is simply because, for whatever reason, there is no further multitude for it to be a member of.

To complete the sketch begun in §3.1, there are topmost multitudes outside the ordinal hierarchy of sets. These are classes, and one can say what they have in common. First, there is no funny business about their membership. Their members are sets, or a mixture of sets and individuals, and nothing else. But, second, for any given level V they each have a member of a level higher than V (see §11, Theorem 41(ii)). It follows that they cannot be assigned an ordinal as rank, since they are not members of any level. We have called classes 'topmost' multitudes, since we do not believe in superordinals beyond the ordinals: once one has exhausted the ordinals, that's it.

It may be helpful to illustrate both these features at work in the altogether simpler context of multitudes of natural numbers. There one has, obviously enough, multitudes bounded above by some number; these are the analogues of our sets. But there are also unbounded multitudes, such as the even numbers, the odd numbers, or the primes, not to mention others whose membership weaves through the numbers in some more elaborate way, but such that for any given n they eventually have a member greater than n. These are the analogues of our classes.

3.3. Ineffability Although one can speak of topmost multitudes i.e. classes, one cannot speak of a topmost *layer* of classes, or a topmost *level*, for this way of speaking is liable to run into trouble. We can put the difficulty in terms of plural description. One can naively define *exhaustive description* as

$$\mathbf{x}\cdot F\mathbf{x} =_{df} \imath\mathbf{y}(M\mathbf{y} \wedge \forall\mathbf{x}(\mathbf{x}\in\mathbf{y}\leftrightarrow F\mathbf{x})$$

That is to say, $\mathbf{x}\cdot F\mathbf{x}$ is the multitude whose members are all and only the items that each separately satisfy F. As the $\imath$ symbol shows, this definition relies on another version of plural description which we here call unique description (see §5.3). Exhaustive description is the version best suited to the case where the predicate in question is distributive. For if F is distributive then many items jointly satisfy F iff each does so separately, so that $\mathbf{x}\cdot F\mathbf{x}$ has a good claim to be read, not just as 'the items that each F', but simply as 'the Fs'.

All this, however, ignores the phenomenon of non-elementhood. Suppose that F holds of some non-element a, as with 'a is not a member of itself', or 'a is a multitude', or 'a is a class'. Then it is true that Fa but not that a is a member of $\mathbf{x}\cdot F\mathbf{x}$, for by hypothesis a is not a member of anything. So the universally quantified clause in the definition of $\mathbf{x}\cdot F\mathbf{x}$ fails, and we discover that despite there being many Fs, '$\mathbf{x}\cdot F\mathbf{x}$' denotes nothing.

The upshot is that classes are *partially ineffable*. Any way of speaking of items that requires classes to be members of multitudes is bound to fail. The ineffability is only partial, however, since we can make definite reference to classes one at a time using unique description, and can also freely generalize about them using quantifier phrases such as 'every class', 'some class', and 'no class', since their correct use does not depend on there being a multitude of all classes.

When talking about multitudes there is bound to be some kind of ineffability, whether or not the full hierarchy of levels is admitted, and whether or not classes are admitted. Suppose first that the hierarchy of levels is curtailed. This may happen in two ways, either by (i) including only those levels below some limit level V_λ. or (ii) including only those levels below some successor level $V_{\alpha+1}$, where V_α itself may be any level, limit or successor. (The bounded and unbounded sets of natural numbers described at the end of §3.2 are a good analogy, with V_λ and V_α both taken to be V_ω.) These curtailments correspond to different restrictions on multitudes, namely to (i) members of levels below V_λ, or (ii) members of levels V_α and below (equivalently, given that levels are cumulative, members of V_α).

In their turn, these restrictions give rise to two broad varieties of ineffability. We can distinguish them by using the naïve version of exhaustive description to explore the different ways in which $\mathbf{x}\cdot F\mathbf{x}$ may be empty. It is convenient to take (ii) first. The restriction to multitudes which are members of some level V_α means that $\mathbf{x}\cdot F\mathbf{x}$ will be empty if V_α is the lowest level of which some F is a member. Since higher levels are excluded, so is any multitude including the relevant F item. As an illustration, consider the obstacle faced by 'first-level' pluralists, who admit only multitudes of individuals. They can make definite reference to such multitudes one at a time. But for them the exhaustive description $\mathbf{x}\cdot F\mathbf{x}$ will be empty if F is ever true of a multitude. There is a corresponding obstacle to some though not all forms of quantification. Thus although they can say 'some multitude has *a* as a member', they cannot say 'some multitudes have a common member', since that would presuppose there being a second-level multitude of the first-level ones in question.

The restriction (i) to multitudes which are members of levels below some limit level V_λ means that $\mathbf{x}\cdot F\mathbf{x}$ will be empty if for every $\alpha<\lambda$, there is some item that is an F whose rank is greater than α, or equivalently, if there is no level below V_λ of which every F is a member. Since V_λ and higher levels are excluded, so is any multitude including every F. Consider for example the obstacle faced by 'finite-level' pluralists, who admit all finite levels but none higher. They can make definite reference to their multitudes one by one. But for them the exhaustive description $\mathbf{x}\cdot F\mathbf{x}$ will be empty if there is no finite level of which every F is a member. There is again a corresponding obstacle to some forms of quantification. The finite-level pluralist can generalize about their multitudes using e.g. 'every finite-level multitude'. But they can only use 'every multitude of finite-level multitudes' in what, from the perspective of the full hierarchy of levels, is a restricted sense. For although they admit some such multitudes, they exclude others, e.g. the multitude of all finite-level multitudes.

Both kinds of restriction mean that there may be no multitude of all Fs despite their being many Fs. Although a particular case of ineffability may be overcome by admitting further levels, the exclusion of any higher ones is bound to generate a new case. How do things stand, then, once *all* levels are admitted? If multitudes are confined to members of levels, i.e. to sets, the effect can be compared to curtailing the hierarchy to levels below some limit level. For now $\mathbf{x}\cdot F\mathbf{x}$ will be empty if there is no level of which every F is a member. This difficulty may be overcome by admitting classes outside the hierarchy, as well as sets within it. The effect of admitting classes can be compared to curtailing the hierarchy to levels below some successor level. As we pointed out above, $\mathbf{x}\cdot F\mathbf{x}$ will now be empty provided some class is F. The virtue of admitting classes is that all the cases of ineffability that can be overcome have been overcome, leaving only this truly inevitable case.

Rayo's stratified variables. In a pioneering piece (2006), Agustín Rayo presented a system of higher-level plural logic modelled on the simple theory of types. His notation was in consequence highly stratified—*x*, *y*, *z*, etc for singular variables, *xx*, *yy*, *zz* for first-level plural variables, *xxx*, *yyy*, *zzz* for second-level ones , and so on. The idea of using repeated letters originates with John Burgess and Gideon Rosen (1997), and has been very influential as far as first-level plural logic is concerned, though one must wonder whether it will be workable in practice for higher levels; see for example this formula from Rayo's 2006, p. 228

$$\exists xxxx(\forall yyy(yyy \prec^{3,4} xxxx \leftrightarrow P^3(yyy)) \wedge \textsc{Ref}^{1,4}\,(`P^3(\ldots)\text{'}, xxxx))$$

Subsequently, however, Rayo's ideas have changed drastically, as may be seen from Linnebo and Rayo (2012); the interested reader will also want to consult Button and Trueman (2022). The type structure is now taken to be cumulative rather than exclusive, infinite types are allowed, and type restrictions on predicates are removed. Each of these changes is carefully argued for, which makes it all the more surprising that Linnebo and Rayo do not even mention a possible fourth change, namely removing type restrictions on terms and variables. Their continued insistence on stratifying variables means than generalizations about sets can only be made for one level at a time, which from our standpoint seems to be massively over-cautious.

§4 The theory of multitudes.

Now we turn to developing a formal theory of multitudes. It draws heavily on Zermelo's idea of a cumulative theory of types (1930) as re-thought by Dana Scott (1974), whose work was in turn improved by John Derrick in unpublished work and by Michael Potter in his book *Set Theory and its Philosophy* (2004). This background, along with the changes to the Scott/Derrick/Potter theory necessitated by the nonexistence of empty and singleton sets, is spelt out in some detail in §5 of our 'Cantorian set theory' (2018).

4.1. Syntax The language of our system is modelled on the familiar language of the predicate calculus with identity, plus the description operator ı and allowing for function signs as well as predicates and constants. Although we talk of 'the' language of our system, what one really has is a family of languages making different selections of non-logical vocabulary to suit different applications.

We opt to take all five principal connectives and the universal quantifier as primitive, but define the existential quantifier. Our variables are 'inclusively' plural in the sense of §1.1; they also may have no value at all (see §5.2). We distinguish them using bold letters **x**, **y**, **z**. We also have a primitive logical predicate ∈ expressing vertical inclusion (membership), to stand alongside the predicate = expressing identity.

We follow convention and call open as well as closed terms simply terms, including variables standing by themselves. As well as constants, terms include descriptions with the form ı**x***A* and 'functional terms' with the form $fa_1 \dots a_n$, obtained by applying a function sign f to its argument terms $a_1 \dots a_n$.

We use *a*, *b*, *c* as schematic letters for terms of arbitrary complexity, including variables standing alone. *A*, *B*, *C* stand for single formulas, and Γ for any number (none or one or more) of formulas. For substitution of terms in formulas, we use the reader-friendly notation $A(a)$ and $A(b)$. When stating the necessary provisos against unintended capture of free variables we supplement the familiar idea of a term's being free for a variable by the notion of a free occurrence of a term. This permits a uniform treatment of substitution of terms for terms, whether variables or not.

Logical vocabulary
Variables **x**, **y**, **z**, countably many
Connectives ¬ → ↔ ∧ ∨, plus brackets for punctuation
Universal quantifier ∀
Unique description operator ı
Identity, a two-place predicate =
Vertical inclusion (aka membership), a two-place predicate ∈

Non-logical vocabulary
Constants
Predicates, each of a specified finite degree
Function signs, each of a specified finite degree

Formation rules
Variables and constants are terms.
If f is an n-place function sign and $a_1 \ldots a_n$ are terms, $fa_1 \ldots a_n$ is a term.
If $\mathbf{x}$ is a variable and A a formula, $\iota\mathbf{x}A$ is a term.
If F is an n-place predicate and $a_1 \ldots a_n$ are terms, $Fa_1 \ldots a_n$ is a formula.
If A and B are formulas, so are $\neg A$, $(A \rightarrow B)$ etc, with the usual conventions for omitting brackets.
If $\mathbf{x}$ is a variable and A a formula, $\forall\mathbf{x}A$ is a formula.

Scope, free and bound occurrences of variables, terms and formulas
The scope of an occurrence of $\forall$ or ι is defined as the shortest formula or term in which it occurs. These operators always occur with a variable attached, as in $\forall\mathbf{x}A$ or $\iota\mathbf{x}A$, and an occurrence of $\mathbf{x}$ is bound if it is within the scope of an operator whose attached variable is $\mathbf{x}$; otherwise it is free. More generally, an occurrence of a term a or formula A in another term or formula is bound if it is within the scope of an operator whose attached variable occurs free in a or A; otherwise it is free.

4.2. Semantics Here we summarise the semantics that underpins our theory, commenting on its principal novelties in the next section.

Individuals
The individuals may be any objects; there may be none or one or more.

Valuation and satisfaction
For each variable $\mathbf{x}$, *val* $\mathbf{x}$ is any item (individual or multitude) or zilch.
For each constant a, *val* a is any item or zilch.
For each n-place predicate F, *val* F is an n-place relation on items, in particular *val* $=$ and *val* $\in$ are the relations of identity and vertical inclusion.
For each n-place function sign f, *val* f is an n-place function on items.

val satisfies $Fa_1 \ldots a_n$ iff *val* F holds of *val* $a_1, \ldots,$ *val* a_n.
val $fa_1 \ldots a_n$ is the value, if any, of *val* f for arguments *val* $a_1, \ldots,$ *val* a_n; otherwise it is zilch.
val satisfies $\neg A$ iff it does not satisfy A. It satisfies $A \rightarrow B$ iff it satisfies B or does not satisfy A. Similarly for the other connectives.
val satisfies $\forall\mathbf{x}A$ iff every $\mathbf{x}$-variant (see §5.3 below) of *val* satisfies A.
val $\iota\mathbf{x}A$ is *val′* $\mathbf{x}$ if a unique $\mathbf{x}$-variant *val′* of *val* satisfies A; otherwise it is zilch.

Logical consequence and logical truth
$\Gamma \vDash C$ iff every valuation, over no matter what individuals (none or one or more), satisfies C if it satisfies every one of Γ. Similarly for $\vDash C$ (logical truth).

§5 Comments on the semantics.

5.1. Items The values of variables are items. An item may be an individual or a multitude. Variations may be obtained by confining items to certain ranks. At the extreme, items are restricted to those of rank 0: this will be a system of singular logic, where a variable can only have an individual as value. Next comes the theory of multitudes of rank at most 1,

… , *n*; then the theory of all multitudes of finite rank, and so on up into the transfinite. These truncated systems can serve as aids for studying the behaviour of sets and classes. For example, when multitudes are confined to those of rank 1, they are all in effect classes, since we are deliberately excluding the higher-level multitudes of which they would otherwise be members. The difference with the unconfined case of §3.3 is simply that in the latter there is nothing beyond classes to be excluded. Similarly, we have seen that the theory of multitudes of rank $\leq \omega$ is a fair simulation of the situation with respect to classes and the uppermost stretch of the hierarchy of sets.

The reader may readily see that what we are here calling a 'theory of multitudes' is merely a notational variant of a system of plural logic. So, for instance, the theory of multitudes of rank at most 1 is the same as the 'full plural logic' of our 2016 (Ch. 13), modulo differences in the choice of primitives. One such difference is that there we take undifferentiated inclusion $\preccurlyeq$, whereas here we have vertical inclusion $\in$ and identity. And where we had two styles of variable, singular and plural, we now only have the latter; but, as we showed (2016, §13.3), all the relevant procedures involving singular variables can be replicated using only plural ones.

5.2. Empty terms A salient feature of the system is its employment of empty terms, that is, terms that denote nothing, aka zilch. Constants, descriptions and functional terms may all be empty. Indeed, no term of any kind necessarily denotes something, since we allow for the possibility that nothing at all exists. In other words, we drop the assumption built into the semantics of the classical predicate calculus that it is logically necessary that something exists. As we have said, we think this violates the topic neutrality that should characterize any logic properly so-called. In the case of variables, and open terms in general, our method for dealing with the possibility that there might be nothing is to permit them, like closed terms, to be empty. So we relax the standard idea of valuation so that variables may either take an item as value or receive no value (be empty). This has the great advantage of settling the logical status of open formulas without disturbing modus ponens.

5.3. Variable-binding Empty variables affect the semantics of open formulas with their free variables but do not affect the semantics of bound variables. Thus the quantification $\forall\mathbf{x}A$ is true iff A is true for every assignment of an item as value of $\mathbf{x}$. When we rephrase this in terms of valuations and satisfaction, we need to take care of the case where the operative variable $\mathbf{x}$ is empty under the given valuation:

> *val* satisfies $\forall\mathbf{x}A$ iff every valuation that assigns a value to $\mathbf{x}$ and differs from *val* at most in that fact and in what that value may be, satisfies A.

The valuations on the right-hand side are thus stipulated to assign a value to $\mathbf{x}$ even if *val* $\mathbf{x}$ is zilch. In the summary in §4.2, we adapted from Mates (1965) the label '$\mathbf{x}$-variant of *val*', now understood as abbreviating 'valuation that assigns a value to $\mathbf{x}$ and differs from *val* at most in that fact and in what that value may be'.

The other primitive variable-binding operator is the unique description operator ι. Its denotation conditions are obtained by generalizing Russell's account of what he calls 'denotation' (1905, p. 51), to allow multitudes as well as individuals to be denoted. Thus if A is true for some unique item as value of $\mathbf{x}$ then $\iota\mathbf{x}A$ denotes that item. If there is no such item, the description is empty. It follows that the semantics of variables bound by ι is not affected by allowing variables to be empty. Just as with quantifiers, we take care of the case where the operative variable is empty under the given valuation by using the $\mathbf{x}$-variant idea.

5.4. Valuations We use 'valuation' or *val* as an umbrella word covering the assignment of values to terms, predicates, and function signs. In each case *val* is a function from linguistic arguments to their semantic values, but the values are very different. For a term a as argument, *val* a is an individual or a multitude or zilch. It is thus a partial function. For an n-place predicate F as argument, *val* F is always an n-place relation (understood as covering properties in the one-place case). For an n-place function sign f as argument, *val* f is always an n-place function.

It is customary to assign sets (or classes) to predicates and function signs as their semantic values. To take the simplest case, a one-place predicate F is assigned a set (or class) such that Fa is true iff the denotation of a is a member of the set (or class). If, as we claim, sets and classes should be regarded as multitudes, can we simply assign the relevant multitude to the predicate? No, for two reasons. In the first place, a predicate may be true of zilch, yet zilch is not a member of any multitude. We shall encounter examples below. And, second, in some cases there is no multitude comprising every multitude a predicate is true of. Examples were given in §3.3. (Note that the problem cases have nothing specially to do with multitudes, since they arise, *mutatis mutandis*, when sets and classes are understood in the orthodox fashion, since e.g. zilch still cannot be a member of them.)

These problems generalize: multitudes will not do as semantic values for n-place predicates or function signs. That is why we have reinstated the relations and functions for which multitudes can at best be artificial surrogates as values, conceiving of them like Frege's *Begriffe* as different from individuals and multitudes, and therefore not values of our first-order variables. It follows that generalizing about 'all valuations' in the metalanguage involves second- (or higher-) order quantification.

5.5. Predication For predication the guiding principle is that

val satisfies $Fa_1 \ldots a_n$ iff *val* F holds of *val* $a_1, \ldots,$ *val* a_n

For satisfaction to be well-defined the right-hand side must always be determinately true or false, whatever the terms $a_1 \ldots a_n$ denote (remembering that they may denote multitudes as well as individuals, or may be empty). Hence we require that the value of an n-place predicate be an n-place relation *on items* in the sense that for any items $\mathbf{x}_1 \ldots \mathbf{x}_n$, the relation either holds or does not hold of $\mathbf{x}_1$ (or zilch), … , $\mathbf{x}_n$ (or zilch) as arguments.

5.6. Strong and weak predicates We say that an n-place predicate F is *strong* at its i-th place if it is necessary for the truth of $Fa_1 \ldots a_n$ that a_i exists; otherwise it is *weak* at that place. When the predicate has just one place, we simply say that it is strong or weak, as the case may be. Primitive non-logical predicates have weak places unless stipulated otherwise. As to the logical predicates = and $\in$, we have opted to make them strong at both places, so that $a{=}b$ and $a{\in}b$ are satisfied only if both a and b exist. A notion of weak identity also proves invaluable, however, and we define it in §6 and discuss it in §9.3.

5.7. Functions, function signs and functional terms Functions and relations are different kinds of beast. Relations *hold of* items, whereas functions *map* arguments *to* values. Relations are as different from functions as they both are from individuals or multitudes. Hence we accept both as their own kind of thing, rather than try reducing one to the other. (This is not to deny that there is an intimate connection between the two: see our 2016, §9.3.)

Just as we allow relations to hold of multitudes as well as individuals, so we allow functions to have multitudes as well as individuals as their arguments or values. This is not the whole story, however. Just as relations may hold of zilch, so there are functions that have no value for certain arguments, in other words, they map those arguments to zilch. These *partial* functions are familiar both in mathematics and everyday life. In our semantics, the function *val* as applied to terms is a case in point. Less familiar are *co-partial* functions, which map zilch to some value, such as *the least natural number that isn't*, which yields 0 when applied to the greatest prime, or $\tan^{-1}$, which maps zilch to all the angles whose cosines are 0. A third possibility is the kind of function that maps zilch to zilch, such as *the moons of* applied to Vulcan. Of course, when there is nothing at all, every function is bound to map zilch to zilch. In our semantics, then, we allow for all three kinds: partial, co-partial, zilch-to-zilch.

As with predicates, we distinguish strong and weak places of function signs. We say that an n-place function sign f is *strong* at its i-th place if it is necessary that if $fa_1 \dots a_n$ exists then so too does a_i, otherwise it is *weak* at that place. When the function sign has just one place, we simply say that it is strong or weak, as the case may be. Thus the function sign '$\tan^{-1}$' is weak, since it expresses a co-partial function, whereas 'the moons of' is strong, since it expresses a zilch-to-zilch function.

For functional terms the guiding principle is

> $val\, fa_1 \dots a_n$ is the value, if any, of $val\, f$ for arguments $val\, a_1, \dots, val\, a_n$; otherwise it is zilch.

In order to ensure that the left-hand side is well-defined, we require that the value of an n-place function sign be an n-place function *on items* in the sense that for any items $\mathbf{x}_1 \dots \mathbf{x}_n$, the function either has some item as value for the arguments $\mathbf{x}_1$ (or zilch), … , $\mathbf{x}_n$ (or zilch), or else has no value for those arguments.

5.8. Logical truth and consequence In interpreting the language it is enough to specify the individuals, since they determine what multitudes there are. Hence the clause 'over no matter what individuals (none or one or more)' in the definitions of logical truth and consequence. Open formulas as well as closed ones can be logically true, e.g. $\vDash A(\mathbf{x}) \vee \neg A(\mathbf{x})$. Indeed, a glance at the elementary axioms in §8.1 reveals that all but one have instances featuring free variables. The exception is Axiom (v) $\forall\mathbf{x}\ \mathbf{x}=\mathbf{x}$. This cannot be replaced by the unquantified $\mathbf{x}=\mathbf{x}$, since none of the latter's instances are logically true. For the strong reading of identity means that $\mathbf{x}=\mathbf{x}$ is not satisfied when $\mathbf{x}$ is empty. Open formulas can also enter relations of logical consequence, e.g. $\mathbf{x}=\mathbf{y} \vDash A(\mathbf{x}) \leftrightarrow A(\mathbf{y})$.

§6. Initial definitions.

Here we comment on some initial definitions before tackling levels in the next section. Rather than put selection restrictions on the eligible arguments of predicates and function signs, we have opted to define them for all terms. So, for instance, $a \subseteq b$ and $a \cap b$ are defined even when a or b denotes an individual or is empty, but in such cases the definitions make $a \subseteq b$ false and $a \cap b$ empty. We have taken as read details about the choice of variables in the definitions. As usual, slashed two-place predicates are convenient shorthand: $a \neq b$ abbreviates $\neg(a=b)$, etc.

As well as defining the existential quantifier in terms of $\forall$, we use identity to define two others.

Existential quantifier $\exists\mathbf{x}A =_{df} \neg\forall\mathbf{x}\neg A$
'Exactly one' quantifier $\exists_1\mathbf{x}A(\mathbf{x}) =_{df} \exists\mathbf{x}\forall\mathbf{y}(A(\mathbf{y})\leftrightarrow\mathbf{x}=\mathbf{y})$
'Many' quantifier $m\mathbf{x}A(\mathbf{x}) =_{df} \exists\mathbf{x}\exists\mathbf{y}(\mathbf{x}\neq\mathbf{y} \wedge A(\mathbf{x}) \wedge A(\mathbf{y}))$

The quantifier $\exists_1$ may be read as 'there is exactly one', or simply 'one', while *m* may be read as 'there are many' or simply 'many', taking it in its weakest sense as equivalent to 'more than one', i.e. 'at least two'.

We use E! to symbolize existence

Existence $\mathsf{E}!a =_{df} \exists\mathbf{x}\ \mathbf{x}=a$

The semantics of ℩ means that $\mathsf{E}!\iota\mathbf{x}A(\mathbf{x})$ is equivalent to $\exists_1\mathbf{x}A(\mathbf{x})$.

As already noted, we have chosen to make = strong at both places. This is embodied in the definition of E!*a* above. But we also define a symbol for weak identity:

Weak identity $a\equiv b =_{df} a=b \vee (\neg\mathsf{E}!a \wedge \neg\mathsf{E}!b)$

The identities $a=b$ and $a\equiv b$ only differ when *a* and *b* are both empty, so we can move freely between them when either or both terms are non-empty. Definitions of terms $a =_{df} b$ permit one to infer the weak $a\equiv b$ but not the strong $a=b$ unless E!*b*.

The 'many' quantifier *m* is used to define the predicate 'are many' symbolised by *M*. Since all and only multitudes are many, *M* may also be read 'is a multitude'

Multitude $Ma =_{df} m\mathbf{x}\ \mathbf{x}\in a$

M is strong by the definition of *m* and the strength of ∈. We use E! and *M* to define the predicate *U*

Individual $Ua =_{df} \mathsf{E}!a \wedge \neg Ma$

U is thus strong too. Together, *U* and *M* provide an exclusive and exhaustive classification of items. Just as *M* is true of all and only multitudes, so *U* is true of all and only individuals. Hence it may be read as 'is an individual'. Since our individuals are the urelements of orthodox set theory, we have opted to use the letter *U*, which also recalls Russell's idea that an individual is a genuine unity, in contrast to a multitude which can only be counted 'as though' it were a genuine unity.

Multitudes are divided into sets and classes using the notion of *element*, symbolized by *E*.

Element $Ea =_{df} \exists\mathbf{x}\ a\in\mathbf{x}$

A set (symbolized by *S*) has members and is a member, whereas a class (symbolized by *C*) has members but is not a member:

Set $Sa =_{df} Ma \wedge Ea$

Class $Ca =_{df} Ma \wedge \neg Ea$

Elements are therefore either individuals or sets.

We define submultitude $\subseteq$ and proper submultitude $\subset$ in the obvious way.

Submultitude $\quad a{\subseteq}b =_{df} Ma \wedge \forall\mathbf{x}(\mathbf{x}{\in}a{\rightarrow}\mathbf{x}{\in}b)$
$a{\subset}b =_{df} a{\subseteq}b \wedge a{\neq}b$

Sets and classes are alike in that they have submultitudes and are themselves submultitudes. If a is a set and a submultitude of b, we say that a is a *subset* of b. If a is a class and a submultitude of b, we say that a is a *subclass* of b. As we shall see, every class has both subsets and subclasses, indeed, infinitely many of each (see §10.7), but the axiom of separation (see 2(iii) in §8.3) means that a set has only subsets, never subclasses.

We symbolize the paradigm empty term 'zilch' by an italic capital O. Although O may be taken as a primitive constant, we opt to define it:

Zilch $\quad O =_{df} \imath\mathbf{x}(\mathbf{x}{\neq}\mathbf{x})$

The description $\imath\mathbf{x}(\mathbf{x}{\neq}\mathbf{x})$ is necessarily empty on account of the logically unsatisfiable condition $\mathbf{x}{\neq}\mathbf{x}$. Hence if F is strong FO is logically false, like $O{=}O$, though when F is weak FO may be true or even logically true, like $O{\equiv}O$ or 'O is a history' in §7. Also $a{\equiv}O$ is equivalent to $\neg\mathsf{E}!a$, and therefore provides another way to express non-existence, and a neat way to say that a function f has no value for an argument a, viz. $fa{\equiv}O$.

We need to emphasize that O does not denote anything whatever, however special or recondite. It denotes zilch, that is, it denotes nothing. Our use of 'zilch' corresponds to the unjustly neglected use of 'nothing' as a necessarily empty term rather than a quantifier. For more see our 2016, pp. 111–14 & 120–28.

Enough has been said to forestall confusion between O and $\varnothing$ as this latter symbol is conventionally understood, namely as standing for the empty set, which is something, not nothing. Zilch and the empty set are alike in that both fail to have members, but the reason in the case of zilch is that it is not even there to have members, which also explains why it is not a member of anything, unlike the empty set. Like $\in$, the predicate $\subseteq$ is strong at both places. So $O{\subseteq}a$ and $a{\subseteq}O$ are bound to be false, whereas orthodox set theory makes the empty set a subset of everything, itself included.

We use $\imath$ to define the notion of *exhaustive* description. There are two versions, exclusively and inclusively plural, symbolized by a raised dot and a colon respectively, with $\mathbf{x}{\cdot}A(\mathbf{x})$ read as 'the elements each satisfying $A(\mathbf{x})$', while $\mathbf{x}{:}A(\mathbf{x})$ is read as 'the element or elements each satisfying $A(\mathbf{x})$'.

Exhaustive description
(exclusive version) $\quad \mathbf{x}{\cdot}A(\mathbf{x}) =_{df} \imath\mathbf{y}(M\mathbf{y}{\wedge} \forall\mathbf{x}(\mathbf{x}{\in}\mathbf{y}{\leftrightarrow}(E\mathbf{x} \wedge A(\mathbf{x}))))$

The definition means that $\mathbf{x}{\cdot}A(\mathbf{x})$ is the multitude comprising all the elements each satisfying $A(\mathbf{x})$; if there is no such multitude, the description is empty. As we shall see in §8.2, Axiom 1(v) governing exhaustive description yields the appropriate comprehension principle: $\mathbf{x}{\cdot}A(\mathbf{x})$ exists just in case there are many elements each satisfying $A(\mathbf{x})$. It is, of course, a further question whether $\mathbf{x}{\cdot}A(\mathbf{x})$ is a set or a class.

Exhaustive description (inclusive version) $\mathbf{x}{:}A(\mathbf{x}) =_{df} \iota\mathbf{y}(\mathbf{y}{=}\iota\mathbf{x}(E\mathbf{x} \wedge A(\mathbf{x})) \vee \mathbf{y}{=}\mathbf{x}{\cdot}A(\mathbf{x}))$

So the descriptions $\mathbf{x}{:}A(\mathbf{x})$ and $\mathbf{x}{\cdot}A(\mathbf{x})$ only differ in denotation when A is satisfied by a unique element. For then the inclusive $\mathbf{x}{:}A(\mathbf{x})$ denotes that element whereas the exclusive $\mathbf{x}{\cdot}A(\mathbf{x})$ is empty. In such a case, the orthodox reading of $\{x{:}A(x)\}$ makes it denote the singleton whose sole member is the element satisfying $A(x)$. Our $\mathbf{x}{:}A(\mathbf{x})$, however, denotes the element itself. To repeat, there are no singleton multitudes.

These definitions are restricted to elements. The reason was given in §3.3, where we saw that the presence of even one non-element among many items satisfying $A(\mathbf{x})$ is enough to destroy the equivalence between '$\mathbf{x}$ is an A' and '$\mathbf{x}$ is one of the As'.

The distinction between two versions of exhaustive description makes it easy to distinguish two kinds of pair, the *proper* pair $|a, b|$, which has the distinctness of a and b built into it, so that it has two members whenever it exists, and the broader notion of the *improper* pair $[a, b]$. Both kinds will be used together to define ordered pairs: see §10.6.

Proper pair $|a, b| =_{df} \mathbf{x}{\cdot}(\mathbf{x}{=}a \vee \mathbf{x}{=}b)$

Improper pair $[a, b] =_{df} \mathbf{x}{:}(\mathbf{x}{=}a \vee \mathbf{x}{=}b)$

$|a, b| \equiv [a, b]$ provided a and b exist and are distinct. The textbooks usually sneak in singletons as a special case of a pair—$\{a\} =_{df} \{a, a\}$—but this trick does not work here, since $|a, a| \equiv O$, while $[a, a] \equiv a$ itself.

The pairing functions expressed by $|\quad|$ and $[\quad]$ are partial. We have already given an example where $|a, b| \equiv O$. Also $|a, b| \equiv [a, b] \equiv O$ when $a \equiv b \equiv O$, or when a and b are classes, since no class is an element. Note that $[a, O] \equiv [O, a] \equiv a$, while $|a, O| \equiv |O, a| \equiv O$.

We also use exhaustive description to define intersection, union, adjunction, power and power-plus operations. Some of the definitions need the inclusive version. For the rest it makes no difference which we use, and we opt for the exclusive version simply because it tends to make proofs smoother.

The inclusive version is used to define the intersection of a and b, symbolized $a{\cap}b$, as the item, if any, that is either their sole common member or the multitude of their many common members.

Intersection $a{\cap}b =_{df} \mathbf{x}{:}(\mathbf{x}{\in}a \wedge \mathbf{x}{\in}b)$

The intersection $a{\cap}b$ may exist when a or b is a class as well as a set, but not when a or b is zilch or an individual, since neither has members.

In the orthodox context, the behaviour of $\cap$ is determined by the equivalence $x{\in}a{\cap}b \leftrightarrow x{\in}a \wedge x{\in}b$. In the present context, the corresponding equivalence $\mathbf{x}{\in}a{\cap}b \leftrightarrow \mathbf{x}{\in}a \wedge \mathbf{x}{\in}b$ holds if a and b have many common members or none, and we highlight the importance of establishing when this is so in our comments on Theorem 5(iii) in §11. But if they have just one common member the left-hand side needs to be replaced by $\mathbf{x}{=}a{\cap}b$, for now $a{\cap}b$ is this item itself, not its members; and experimentation soon produces counterexamples to the associativity of $\cap$. For instance, let a and b be disjoint sets and c be $|a, b|$, and compare $|a, c| \cap (|a, c| \cap |b, c|)$ with $(|a, c| \cap |a, c|) \cap |b, c|$. The former is $|a, c| \cap \mathrm{c} = |a, c| \cap |a, b| = a$, while the latter is $|a, c| \cap |b, c| = c$. This deviation is no more surprising than the failure of ordinal addition to be commutative, and it turns out to be insignificant in practice, since throughout the entire development described in §11 and the Appendix, although we work with iterated applications of intersection, the absence of associativity is never an obstacle.

For a generalized version of intersection we reuse $\cap$ as a variable-binding operator forming terms from formulas.

Generalized intersection $\cap\mathbf{x}A(\mathbf{x}) =_{df} \mathbf{x}{:}(\forall\mathbf{y}(A(\mathbf{y})\rightarrow\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{y}A(\mathbf{y}))$

Thus $\cap\mathbf{x}A(\mathbf{x})$ may exist even when there is just one item satisfying $A(\mathbf{x})$ or when a class satisfies $A(\mathbf{x})$. We use $\cap a$ to abbreviate $\cap\mathbf{x}\ \mathbf{x}\in a$, symbolizing the case where $A(\mathbf{x})$ expresses membership in some multitude.

The union of a and b, symbolized $a\cup b$, is defined to be the multitude, if any, of items that are members of a or of b.

Union $a\cup b =_{df} \mathbf{x}{\cdot}(Ma \wedge Mb \wedge (\mathbf{x}\in a \vee \mathbf{x}\in b))$

Like the intersection $a\cap b$, the union $a\cup b$ may exist when a or b is a class as well as a set, but not when a or b is zilch or an individual. We have used the exclusive $\cdot$ to define $a\cup b$, but the inclusive : would do equally well, since $a\cup b$, if it exists, is always a multitude.

To obtain a generalized version of union, we reuse $\cup$ as a variable-binding operator.

Generalized union $\cup\mathbf{x}A(\mathbf{x}) =_{df} \mathbf{x}{\cdot}(\forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{y}(A(\mathbf{y}) \wedge \mathbf{x}\in\mathbf{y}))$

As with generalized intersection, $\cup\mathbf{x}A(\mathbf{x})$ may exist even when there is just one item satisfying $A(\mathbf{x})$ or when a class satisfies $A(\mathbf{x})$. We use $\cup a$ to abbreviate $\cup\mathbf{x}\ \mathbf{x}\in a$, thus symbolizing the case where $A(\mathbf{x})$ expresses membership in some multitude.

In orthodox set theory, the result of adding an element to a set (or class) is the union of the singleton of the element with the set (or class). But we can obtain the same result without singletons by defining an adjunction operation, symbolized $a\oplus b$ (see Bernays 1937, p. 68).

Adjunction $a\oplus b =_{df} \mathbf{x}{\cdot}(Ma \wedge Eb \wedge (\mathbf{x}\in a \vee \mathbf{x}=b))$

As with union, the inclusive : would do equally well here.

Next comes the power multitude of a, symbolized $P(a)$, which may be read 'the subset or subsets of a'

Power $P(a) =_{df} \mathbf{x}{:}\mathbf{x}\subseteq a$

When a is an individual or zilch, $P(a)$ is empty, since neither has subsets. When a is a multitude, $P(a)$ is non-empty, since any multitude has at least one subset. Note that if a is a class, only its subsets get into $P(a)$, since its subclasses are not elements. Also, if a is a set with exactly two members, a has just one subset, namely itself. We have found it marginally more convenient to define $P(a)$ inclusively, so that in this case $P(a)=a$; the alternative exclusive definition would render $P(a)$ empty.

Finally, the power-plus multitude of a, symbolized $P^{+}(a)$. This is defined as the members and subsets of a.

Power-plus $P^{+}(a) =_{df} \mathbf{x}{\cdot}(\mathbf{x}\in a \vee \mathbf{x}\subseteq a)$

Like $P(a)$, $P^{+}(a)$ is non-empty just in case a is a multitude, and when a is a class, only its members and subsets belong to $P^{+}(a)$. But unlike $P(a)$, using the inclusive : to define $P^{+}(a)$ would make no difference. For the formula embedded within the relevant description will be multiply satisfied if it is satisfied at all, since only multitudes have members or subsets, and any multitude has many members.

§7 Levels.

Recall the guiding idea that a level is the set comprising all the individuals plus the members and subsets of all lower levels, or for short, a level is *the accumulation of its history*, where the history of a level is the level(s), if any, lower than it. In §3.1 we used ordinals to impose a well-ordering on levels and thereby give a clear meaning to 'lower level'. But inside the system we can use the well-ordering provided by $\in$ itself. We reserve the variables **u**, **v**, **w** for levels. Then **u** is lower than **v** if $\mathbf{u}\in\mathbf{v}$. The history of any level **v** can be neatly covered by the inclusive description $\mathbf{w}{:}\mathbf{w}\in\mathbf{v}$.

Rather than take 'level' as primitive and rely on an axiom to ensure that a level is the appropriate accumulation, we give a definition of 'level' that makes the axiom redundant. (This was John Derrick's idea. See our 2016: 276.) The key is a prior definition of *history* which does not presuppose the notion of level but ensures that the history of a level **v** is $\mathbf{w}{:}\mathbf{w}\in\mathbf{v}$. It uses two versions of accumulation. We symbolize the first by acc:

Acc function $\quad \mathrm{acc}(a) =_{df} \mathbf{x}\cdot(U\mathbf{x} \vee \exists\mathbf{y}(\mathbf{y}\in a \wedge (\mathbf{x}\in\mathbf{y} \vee \mathbf{x}\subseteq\mathbf{y})))$

This delivers the right result for any level higher than V_2, for then the relevant a is the multitude of many lower levels. But it fails for V_2 with its sole lower level V_1, for the condition $\mathbf{y}\in a$ picks out the members of V_1, not V_1 itself. Since V_1 comprises all the individuals, and nothing is a member or submultitude of any individual, the result is that acc $V_1 = \mathbf{x}\cdot U\mathbf{x} = V_1$, not V_2. We therefore introduce another version, obtained by replacing $\mathbf{y}\in a$ by $\mathbf{y}=a$ in the definition of acc, which then simplifies to

Accum function $\quad \mathrm{accum}(a) =_{df} \mathbf{x}\cdot(U\mathbf{x} \vee \mathbf{x}\in a \vee \mathbf{x}\subseteq a)$

V_2 can now be defined to give the desired result:

Second level $\quad V_2 =_{df} \mathrm{accum}(V_1)$

This leaves V_1, whose history is zilch. Again we use accum to define it:

First level $\quad V_1 =_{df} \mathrm{accum}(O)$

Although we introduced them separately, it is worth remarking that acc can be defined in terms of accum as follows:

$$\mathrm{acc}(a) =_{df} \mathbf{x}\cdot(U\mathbf{x} \vee \exists\mathbf{y}(\mathbf{y}\in a \wedge \mathbf{x}\in\mathrm{accum}(\mathbf{y})))$$

Both functions may take any item or zilch as argument, but the corresponding value, if any, is always a multitude. If there is but one individual, both map it to zilch, that is to say, they are partial.

The definitions of V_1 and V_2 illustrate the utility of exclusive exhaustive description. We want them, like all levels, to be sets if they exist at all. But if inclusive : is used in defining accum, then when there is but one individual $\mathbf{x}$, $V_1 = \text{accum}(O) = \mathbf{x}$, but at the same time $V_2 = \text{accum}(V_1) = \text{accum}(\mathbf{x}) = \mathbf{x}$.

Next we define 'is a history', symbolized by *H,* without invoking the notion of level:

History $$Ha =_{df} (a{\equiv}O \vee Sa) \wedge (a{=}V_1 \vee \forall\mathbf{x}(\mathbf{x}{\in}a{\rightarrow}(\mathbf{x}{=}V_1 \vee \mathbf{x}{=}V_2 \rightarrow \mathbf{x}{=}\text{accum}(a{\cap}\mathbf{x})) \wedge (\mathbf{x}{\neq}V_1 \wedge \mathbf{x}{\neq}V_2 \rightarrow \mathbf{x}{=}\text{acc}(a{\cap}\mathbf{x}))))$$

This needs a little explanation. We reserve the variable $\mathbf{h}$ for histories. If $V_1{=}\text{accum}(\mathbf{h})$, we say that V_1 has $\mathbf{h}$ as a history; similarly for V_2. If $\mathbf{v}$ is any level other than V_1 or V_2, and $\mathbf{v}{=}\text{acc}(\mathbf{h})$, we say that $\mathbf{v}$ has $\mathbf{h}$ as a history. We want any level $\mathbf{v}$ to have $\mathbf{w}{:}\mathbf{w}{\in}\mathbf{v}$ as its unique history. But uniqueness fails if an individual $\mathbf{x}$ counts as a history. For supposing V_1 exists, $V_1 = \text{accum}(O)$ and also $V_1 = \text{accum}(\mathbf{x})$. To rule this out, the first conjunct in the definition requires that a history, if it exists at all, be a *set*. We cannot weaken this to *multitude*, for then the class of all levels would count as a history, which runs counter to the idea that histories are histories of levels, since there is no level whose lower levels comprise all levels. The first conjunct in the definition also counts zilch as a history: it is the history of V_1, which has no lower levels. Consequently, H is a weak predicate.

As to the second conjunct, naturally we deal separately with the possibility that $\mathbf{x}$ is V_1 or V_2 by replacing acc by accum for these cases. But we also need to cover the possibility that a is V_1. The problem is the same as before. Every member of V_1 is an individual and is therefore neither V_1 nor V_2. But no individual $\mathbf{x}$ is $\text{acc}(a{\cap}\mathbf{x})$, since acc only has multitudes as values.

We have now assembled all the materials needed for the definition of 'is a level', symbolized by V. It should be no surprise that the exceptional levels V_1 and V_2 are mentioned separately, while the rest can be characterized as $\text{acc}(\mathbf{x})$ for some history $\mathbf{x}$

Level $$Va =_{df} a{=}V_1 \vee a{=}V_2 \vee \exists\mathbf{x}(H\mathbf{x} \wedge a{=}\text{acc}(\mathbf{x}))$$

We next define two functions, V^* and $V^\dagger$, which mark the location of elements within the hierarchy of levels using the salient relations of subset and membership. $V^*(a)$ is the lowest level of which a is a subset (*the level of a*, for short). $V^\dagger(a)$ is the lowest level of which a is a member.

V^* $$V^*(a) =_{df} \iota\mathbf{x}(V\mathbf{x} \wedge a{\subseteq}\mathbf{x} \wedge \neg\exists\mathbf{y}(V\mathbf{y} \wedge \mathbf{y}{\in}\mathbf{x} \wedge a{\subseteq}\mathbf{y}))$$

$V^\dagger$ $$V^\dagger(a) =_{df} \iota\mathbf{x}(V\mathbf{x} \wedge a{\in}\mathbf{x} \wedge \neg\exists\mathbf{y}(V\mathbf{y} \wedge \mathbf{y}{\in}\mathbf{x} \wedge a{\in}\mathbf{y}))$$

There are three exclusive and exhaustive kinds of level: the first level, levels next above a level, and limit levels. We have already defined V_1. The level next above a, symbolized by a', is defined to be the lowest level that is higher than the level a, that is to say, the lowest level of which a is a member:

Level next above $a' =_{df} \iota\mathbf{x}(Va \wedge \mathbf{x}{=}V^{\dagger}(a))$

The definition means that $a' \equiv V^{\dagger}(a)$ provided a is a level; otherwise a' is empty.

Finally, we define 'a is a limit level', symbolized by La, to mean that a is a level not of the first two kinds, that is to say, neither the first nor next above any level.

Limit level $La{=}_{df}\ Va \wedge a{\neq}V_1 \wedge \neg\exists\mathbf{x}\ a{=}\mathbf{x}'$

§8 Axioms.

In this section we present an axiomatic system for plural logic with modus ponens as the sole rule of inference. The axioms are all the instances of the following schemes both as they stand and prefaced by any number of universal quantifications. After the elementary axioms, we present the remaining ones in two groups, according as they govern membership or levels.

8.1. Elementary The elementary axioms pluralize the axioms given in §1.2 for a topic-neutral version of the predicate calculus, by substituting plural variables for singular ones throughout.

(i) A where A is tautologous
(ii) $\forall\mathbf{x}(A{\rightarrow}B) \rightarrow (\forall\mathbf{x}A \rightarrow \forall\mathbf{x}B)$
(iii) $A \rightarrow \forall\mathbf{x}A$ where $\mathbf{x}$ is not free in A
(iv) $\forall\mathbf{x}A(\mathbf{x}) \rightarrow (\mathsf{E}!a \rightarrow A(a))$ where $A(a)$ has free a wherever $A(\mathbf{x})$ has free $\mathbf{x}$
(v) $\forall\mathbf{x}\ \mathbf{x}{=}\mathbf{x}$
(vi) $a{=}b \rightarrow (A(a) \leftrightarrow A(b))$ where $A(b)$ has free b at zero or more places where $A(a)$ has free a
(vii) $(\neg\mathsf{E}!a \wedge \neg\ \mathsf{E}!b) \rightarrow (A(a) \leftrightarrow A(b))$ where $A(b)$ has free b at zero or more places where $A(a)$ has free a
(viii) $a{=}b \rightarrow \mathsf{E}!a \wedge \mathsf{E}!b$
(ix) $\forall\mathbf{y}(\mathbf{y}{=}\iota\mathbf{x}A \leftrightarrow \forall\mathbf{x}(A \leftrightarrow \mathbf{x}{=}\mathbf{y}))$ where $\mathbf{y}$ does not occur in $\iota\mathbf{x}A$

8.2. Group 1: Membership Axiom 1(i) ensures that $\in$ is strong at both places. 1(ii) is extensionality for multitudes. 1(iii) ensures that $\in$ is irreflexive, while 1(iv) rules out singletons, i.e. there is no $\mathbf{y}$ such that $\mathbf{x}{\in}\mathbf{y}$ for just one $\mathbf{x}$. It also, via the definition of U, prevents individuals having members. 1(v) governs the exclusive version of exhaustive description, and together with 1(i) gives a principle of comprehension: the multitude $\mathbf{x}{\cdot}A(\mathbf{x})$ exists provided there are many elements that are each A.

1(i) $a{\in}b \rightarrow \mathsf{E}!a \wedge \mathsf{E}!b$
1(ii) $Ma \wedge Mb \rightarrow \forall\mathbf{x}(\mathbf{x}{\in}a{\leftrightarrow}\mathbf{x}{\in}b) \rightarrow a{=}b$ where $\mathbf{x}$ is not free in a or b
1(iii) $a{\notin}a$
1(iv) $\exists\mathbf{x}\ \mathbf{x}{\in}a \rightarrow Ma$
1(v) $\forall\mathbf{y}(\mathbf{y}{\in}\mathbf{x}{\cdot}A(\mathbf{x}) \leftrightarrow m\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z})) \wedge E\mathbf{y} \wedge A(\mathbf{y}))$ where $A(\mathbf{z})$ and $A(\mathbf{y})$ have free $\mathbf{z}$ and free $\mathbf{y}$, respectively, wherever $A(\mathbf{x})$ has free $\mathbf{x}$

8.3. Group 2: Levels Axiom 2(i) ensures that if many individuals exist, then each is an element, from which, together with other axioms, it follows that the first level V_1 exists. 2(ii) says that there are no multitudes, and hence no levels, unless there are many individuals. 2(iii) is modelled on Cantor's own principle of separation, 'every submultitude of a set is a set' (1932, p. 444). 2(iv) restricts sets to the hierarchy of levels but without assuming that there are any. In the context of the other axioms, the converse of 2(iv) follows from 2(iii), with the result that a multitude is a set just in case it is a submultitude of a level. 2(v) says that for every level there is a higher one—from which it follows that there are infinitely many levels, if there are any at all. 2(vi) says that a limit level exists if there are any levels at all. Both the notation and the significance of 2(vii), the Axiom of Ordinals, are explained below. We have not explored its implications for the independence of the other axioms.

2(i) $m\mathbf{x}U\mathbf{x} \to \forall\mathbf{x}(U\mathbf{x}\to E\mathbf{x})$
2(ii) $\exists\mathbf{x}M\mathbf{x} \to m\mathbf{x}U\mathbf{x}$
2(iii) $Sa \wedge b\subseteq a \to Sb$
2(iv) $Sa \to \exists\mathbf{x}(V\mathbf{x} \wedge a\subseteq\mathbf{x})$ where $\mathbf{x}$ does not occur in a
2(v) $\forall\mathbf{x}(V\mathbf{x} \to \exists\mathbf{y}(V\mathbf{y} \wedge \mathbf{x}\in\mathbf{y}))$ where $\mathbf{x}$ and $\mathbf{y}$ are distinct
2(vi) $\exists\mathbf{x}V\mathbf{x} \to \exists\mathbf{x}L\mathbf{x}$
2(vii) $\forall\mathbf{x}(\text{ord }\mathbf{x} \to \exists\mathbf{y}(V\mathbf{y} \wedge \rho(\mathbf{y})=\mathbf{x}))$ where $\mathbf{x}$ and $\mathbf{y}$ are distinct

8.4. Axiom of Ordinals In §3.1 we introduced the rank of a set as an ordinal measure of its height in the hierarchy of levels of plurality. The problem now is to represent the notions of ordinal and rank within the formal theory, so as to be able to formulate a suitable axiom reflecting the height of the hierarchy itself. Our axiom is a simplified analogue of the more specific 'axiom of ordinals' presented in Potter's 2004. Our account is necessarily highly compressed, and the reader is recommended to consult §§4.4, 4.10, 11.1, 11.2, 11.5, 13.2 of Potter's book, while taking into account the adjustments sketched below, which are needed to suit the present context.

Ordinals are commonly introduced as the order-types of well-ordered sets, but to avoid ambiguity a well-ordered set will be represented by a *structure*, an ordered pair whose first coordinate is the set and whose second is a well-ordering on it, the relation being represented by a set of ordered pairs. In order to provide for structures whose ordinals are 0 and 1, we allow the first coordinate of structures to be zilch or an individual; they are both well-ordered by zilch considered as a relation. As to structures whose ordinal is 2, the first coordinate is a pair set and the well-ordering relation is a single ordered pair, not a set of pairs. These adjustments are necessary in the absence of empty and singleton sets.

Our surrogate ordinal of such a structure would be expected to be the multitude of all structures isomorphic to it, with a slight tweaking of the definition of isomorphism to take account of the newly admissible coordinates in structures. But we follow Potter in using Scott's 'trick' (Potter 2004 §4.4, Scott 1955) to ensure that the result is always a set. For any structure a, there is a unique lowest level $\mathbf{v}$ which has as a subset some structure isomorphic to a. Then ord a can be defined as the set of structures isomorphic to a whose level is $\mathbf{v}$ (the exception is the structure $<O, O>$ which will be its own ordinal, viz. the ordinal 0).

In the general case, the rank $\rho(\mathbf{x})$ of a set can now be represented by the ordinal

$$\rho(\mathbf{x}) =_{df} \text{ ord } \langle \mathbf{v}\cdot\mathbf{v}\in V^{\dagger}(\mathbf{x}), \subseteq\rangle$$

where $V^{\dagger}(\mathbf{x})$ is the lowest level of which **x** is a member, as introduced in §7, and $\subseteq$ is an ordered pair or set of ordered pairs as the case may be, representing the restriction of the submultitude relation to the relevant levels. The exception is when $V^{\dagger}(\mathbf{x})$ is the second level V_2, in which case $\rho(\mathbf{x}) =_{df} 1$.

Our axiom 2(vii) says, then, that for every ordinal there is a level whose rank is that ordinal. Potter (2004, §§11.5, 13.2) goes a step further and defines for each ordinal a particular corresponding level as candidate for the role.

Commenting on his axiom, Potter points out that the existence of the new levels which it entails ensures the existence of further ordinals that were not previously available, and the axiom applied to *them* guarantees the existence of corresponding levels; and so on. 'The hierarchy described by the new theory is therefore colossally higher than anything we could have countenanced before' (2004, p. 218).

Since our concern is with the conceptual question about the nature of sets and classes, our development of the theory is limited to what is needed to put the general theory of levels on a proper footing, and to enunciate some results about sets and classes that will hold good independently of issues about higher infinities. We happily leave the investigation and application of the Axiom of Ordinals to any interested reader.

§9 Developing the system: options.

9.1. Incompleteness Any axiomatization of plural logic is necessarily incomplete. This can be shown in the same way as the unaxiomatizability of second-order logic, namely by formulating a version of Peano arithmetic with a finite number of axioms. The resulting theory is categorical and therefore complete, whence by Gödel's theorem it follows that the logical truths of the underlying logic are not effectively enumerable. In each case the key step is the replacement of the axiom scheme for induction by a single axiom, achieved here as follows

$$\forall\mathbf{x}(\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow U\mathbf{y}) \wedge 0\in\mathbf{x} \wedge \forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow s\mathbf{y}\in\mathbf{x}) \rightarrow \forall\mathbf{y}(U\mathbf{y}\rightarrow\mathbf{y}\in\mathbf{x}))$$

In English: if some individuals include zero and also the successor of any of them, every individual is one of them, i.e. the individuals are all and only the natural numbers.

9.2. Axiom of Choice Any added axioms are likely to be plural analogues of ones already in circulation in a singular form. The Axiom of Choice is a good illustration of the considerations involved. In one standard version it runs

> For every nonempty set *x* of nonempty, pairwise disjoint sets, there exists a 'choice' set whose intersection with each member of *x* is a singleton.

The first and most important step is to replace all reference to sets understood as individuals by sets understood as multitudes. There is no need to repeat the requirement that all the relevant sets are nonempty, since that is now a given. But the treatment of intersection calls for the policy of 'judicious replacement' of §2.2. Where sets *y* and *z* were said to be disjoint if $y \cap z = \varnothing$, the analogous condition is $\mathbf{y} \cap \mathbf{z} \equiv O$, and where two sets share a unique member it is that common member itself, not its singleton, that is their intersection. Lastly, there is always the question whether the plural version must also be confined to sets, or whether it can be applied to multitudes in general. For this may not be a simple matter of strength. Obviously 'For any set … there is a choice set' does not imply 'For any multitude … there is a choice multitude', but equally the latter does not trivially imply the former. And there may or may not be a place for a 'class' version of the axiom, viz. 'For every class … there is a choice class'.

9.3. Weak identity Our system accommodates two variant notions of identity, differing according to the truth-value of an identity statement when both its terms are empty. Strong identity, symbolized by =, makes $a=b$ false if either or both terms are empty. Weak identity, symbolized by ≡, agrees in making $a \equiv b$ false if just one of a and b are empty, but makes it true if both are empty. (For a fuller discussion, see our 2016, §7.2.) The two relations are interdefinable in an almost symmetrical way

$$a \equiv b =_{df} a = b \vee (\neg \mathsf{E}!a \wedge \neg\ \mathsf{E}!b) \qquad\qquad a = b =_{df} a \equiv b \wedge (\mathsf{E}!a \wedge \mathsf{E}!b)$$

where in the second definition $\mathsf{E}!a$ is defined as $\exists \mathbf{x}\ \mathbf{x} \equiv a$ instead of $\exists \mathbf{x}\ \mathbf{x} = a$. If we choose to take ≡ rather than = as primitive, then the first three 'elementary' axioms of §8.1 stay as they are, while the remaining six, which involve = either explicitly or implicitly via $\mathsf{E}!$, are replaced by

(iv)′ $\forall \mathbf{x} A(\mathbf{x}) \to (\mathsf{E}!a \to A(a))$
(v)′ $\forall \mathbf{x}\ \mathbf{x} \equiv \mathbf{x}$
(vi)′ $\forall \mathbf{x}\ \mathsf{E}!\mathbf{x}$
(vii)′ $a \equiv b \to (A(a) \leftrightarrow A(b))$
(viii)′ $(\neg \mathsf{E}!a \wedge \neg \mathsf{E}!b) \to a \equiv b$
(ix)′ $\forall \mathbf{y}(\mathbf{y} \equiv \iota \mathbf{x} A \leftrightarrow \forall \mathbf{x}(A \leftrightarrow \mathbf{x} \equiv \mathbf{y}))$

All the new axioms carry the same technical provisos as those they replace. Axiom (iv) is replaced by the visual replica (iv)′, but where $\mathsf{E}!a$ is now defined as $\exists \mathbf{x}\ \mathbf{x} \equiv a$. Axioms (v)′ and (vi)′ together are equivalent to the old (v), while (vii)′ is equivalent to the old (vi) and (vii) taken together. Axiom (viii) is replaced by (viii)′, while (ix)′ is obtained from (ix) by replacing = by ≡.

There are no considerations of economy or convenience in favour of taking = rather than ≡ as primitive, and our decision to do so was motivated by nothing more than a desire to make as few changes as possible to the familiar format of the predicate calculus. So far, so unremarkable, but it turns out that the matter has an altogether greater significance, as we now explain.

9.4. Ineliminability of definite description The reader may have wondered why we have taken the definite description operator ι as primitive. Surely Russell long ago demonstrated the contextual eliminability of ι, starting with the

Basic equivalence $F(\iota\mathbf{x}A) \leftrightarrow \exists_1\mathbf{x}A \wedge \forall\mathbf{x}(A \to F\mathbf{x})$

where F is any primitive predicate, and appealing to the substitutivity of equivalents to produce an ι-less match for any formula.

The presence of a weak primitive predicate, however, demolishes this argument by undercutting its premise. For take any case where nothing satisfies A. Then certainly $\exists_1\mathbf{x}A$ is false and with it the right-hand side of the basic equivalence. Also, the definite description $\iota\mathbf{x}A$ is now an empty term, but if F is weak it no longer follows that $F(\iota\mathbf{x}A)$ is false. In short the left-hand side of the basic equivalence may be true while the right-hand side is false, making the equivalence itself fail. Indeed, by taking Fa to be $a \equiv a$ we can sharpen this result, for now there is no 'may be' about the failure of the basic equivalence, since as a matter of logic its left-hand side is always true.

The outcome, then, is that descriptions, and more generally function signs, are only eliminable if empty terms are ruled out of consideration, and it is a great pity that both the founding fathers of the predicate calculus, Frege and Hilbert, would go to any lengths to avoid dealing with them.

§10 Developing the system: topics.

Here we give a prose overview of the topics covered in our development of the system, referring forward to the relevant results stated in §11.

10.1. Well-ordering of levels The key property of a history is that membership is well-founded on it (Theorem 7). It follows that levels are transitive and hereditary sets: any member or subset of a member of a level is also a member of that level (Theorems 8 and 9). Membership between levels is a transitive relation by the transitivity of levels, and irreflexive by Axiom 1(iii). It can also be shown that membership between levels is well-founded (Theorem 13), and that levels are comparable under membership (Theorem 14). Hence membership well-orders levels. So too does proper submultitude, since the two relations are equivalent among levels (Theorem 27).

10.2. History and composition of levels A level has the levels below it, if any, as its unique history (Theorem 16). Since V_1 is the lowest level of all, its history is zilch. The history of V_2 is the single level V_1, while the history of any other level is the multitude of the many levels below (Theorem 10). V_1 exists just in case there are many individuals. It is the multitude of all the individuals (Theorem 29). For any level $\mathbf{v}$, there is a unique level $\mathbf{v}'$ next above $\mathbf{v}$, which can be shown to be the power-plus multitude of $\mathbf{v}$. Equivalently, $\mathbf{v}'$ is the multitude of all individuals plus all the submultitudes of $\mathbf{v}$ (Theorems 30 and 31). Finally, a level is a limit level just in case it is the union of the levels below, that is to say, the union of its history (Theorem 34). So, for example, the lowest limit level V_ω is the union of all the finite levels.

10.3. Elements and levels An element's location with respect to the hierarchy of levels can be characterised using the salient relations of membership and subset. Every individual is a member of every level (Lemma 8(iv)) and by definition a subset of none. Every set is a member of some level (Theorem 22(i)), and also a subset of some level (Axiom 2(iv)). Since levels are transitive and hereditary, if a set bears either relation to a level, it bears it to every higher level. Transitivity also implies that if a set is a member of some level, it is a proper subset of it too. But not always vice versa, as the following shows. For every set **x** there is a unique lowest level $V^*(\mathbf{x})$—the level of **x**—of which it is a subset (Theorem 17(i)). Also, for every set **x** there is a unique lowest level $V^\dagger(\mathbf{x})$ of which it is a member. It is not the level of **x**, however, but the level next above (Theorem 32).

10.4. Derived principles for multitudes and sets Comprehension is a straightforward consequence of Axiom 1(v) governing the exclusive version of exhaustive description.

Comprehension	$m\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z})) \rightarrow M(\mathbf{x}\cdot A(\mathbf{x}))$	Lemma 2(i)

Comprehension readily yields the following principles for multitudes:

Union	(i) $M\mathbf{x} \wedge M\mathbf{y} \leftrightarrow M(\mathbf{x}\cup\mathbf{y})$	Theorem 20(i)
	(ii) $(\forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z})) \leftrightarrow M(\cup\mathbf{x}A(\mathbf{x}))$	Theorem 21(i)
Pairing	$(E\mathbf{x} \wedge E\mathbf{y} \wedge \mathbf{x}\neq\mathbf{y}) \leftrightarrow M\|\mathbf{x}, \mathbf{y}\|$	Theorem 23(i)
Adjunction	$M\mathbf{x} \wedge E\mathbf{y} \leftrightarrow M(\mathbf{x}\oplus\mathbf{y})$	Theorem 24(i)
Power	$M\mathbf{x} \leftrightarrow M(P(\mathbf{x}))$	Theorem 25(iii)
Power-plus	$M\mathbf{x} \leftrightarrow M(P^+(\mathbf{x}))$	Theorem 26(i)

Like any other multitude, sets exist when and only when there are many individuals (Theorems 28(i) and 28(iii)). For sets, Comprehension has been replaced by the weaker principle of Separation, and for us this means Cantor's own version enunciated as Axiom 2(iii) in §8.3, in which the separated items must be many. Since levels are sets, it follows by Separation that to prove that a multitude is a set it suffices to find a level of which it is a submultitude. This strategy of *separating from levels* yields the following principles for sets:

Union	(i) $S\mathbf{x} \wedge S\mathbf{y} \leftrightarrow S(\mathbf{x}\cup\mathbf{y})$	Theorem 20(ii)
	(ii) $(\forall\mathbf{y}(A(\mathbf{y})\rightarrow S\mathbf{y}) \wedge S(\mathbf{z}{:}A(\mathbf{z}))) \leftrightarrow S(\cup\mathbf{x}A(\mathbf{x}))$	Theorem 21(ii)
Pairing	$(E\mathbf{x} \wedge E\mathbf{y} \wedge \mathbf{x}\neq\mathbf{y}) \leftrightarrow S\|\mathbf{x}, \mathbf{y}\|$	Theorem 23
Adjunction	$S\mathbf{x} \wedge E\mathbf{y} \leftrightarrow S(\mathbf{x}\oplus\mathbf{y})$	Theorem 24(ii)
Power	$S\mathbf{x} \leftrightarrow S(P(\mathbf{x}))$	Theorem 25(ii)
Power-plus	$S\mathbf{x} \leftrightarrow S(P^+(\mathbf{x}))$	Theorem 26(ii)

We can also derive

Foundation	$M\mathbf{x} \rightarrow \exists\mathbf{y}(\mathbf{y}\in\mathbf{x} \wedge \mathbf{x}\cap\mathbf{y}\equiv O)$	Theorem 18

which says that any multitude has a member whose intersection with that multitude is zilch. It is a general version of the familiar foundation (regularity) principle, covering both sets and classes. It follows from a narrower foundation principle governing levels to the effect that there is a lowest among the levels included in any multitude (Theorem 13), which in turn follows from the well-foundedness of membership on any history (Theorem 7).

10.5. Representing the natural numbers We define a to be *inductive* if (i) V_1 is a member of a and (ii) if a level is a member of a, so is the level next above. Given the Axiom of Plurality, the lowest limit level V_ω exists and is inductive (Theorems 34(i), 35 and 36). It follows that there is an inductive set **N*** comprising the common members of every inductive set (Theorem 37). This intersection **N*** is the set of finite levels. With the first natural number

0 (the cardinal and ordinal number of zilch) defined as the first level V_1 and the successor of a level defined as the level next above, it is straightforward to derive Peano's axioms.

10.6. Ordered pairs Our definition of ordered pairs is obtained by reworking Hausdorff's technique, using the first two levels as markers as well as both proper and improper pair sets (for more see our 2018, §9).

Ordered pair $<a, b> =_{df} |||[a, V_1], [a, V_2]|, V_1|, ||[b, V_1], [b, V_2]|, V_2||$

Ordered pairs exist and have their so-called characteristic property, provided that there are many individuals, and also provided that the coordinates of the putative pairs are not classes (Theorem 38). A coordinate may thus be an individual or a set or even zilch. Our version of Hausdorff is more complicated than it otherwise would have been, since we have aimed for greater versatility, allowing O as well as the markers V_1 and V_2 to feature as coordinates of ordered pairs. We could go on to define a still wider notion of ordered pair $<<a, b>>$ to allow for classes as coordinates, even though classes cannot be members of *unordered* pairs. A neat way of achieving this is to define $<<a, b>>$ for sets or classes a, b as the 'disjoint union' of a and b, that is to say, the multitude of all ordered pairs of the form $<\mathbf{x}, V_1>$ or $<\mathbf{y}, V_2>$ where $\mathbf{x}\in a$ and $\mathbf{y}\in b$ (our version of Rubin's definition in her 1967, p. 155). The use of the levels V_1 and V_2 as markers ensures that the disjoint union uniquely determines the coordinates a and b and their order, as required. This definition can easily be extended to cover individuals and zilch as coordinates too. And the definitions of both $<a, b>$ and $<<a, b>>$ can be naturally generalized to cover ordered n-tuples, by bringing in further finite levels as markers, V_1, V_2, …, V_n.

10.7. Classes There is more than one orthodox system that admits classes, notably Von Neumann–Bernays–Gödel set theory and Morse–Kelley set theory. The latter, however, is the only system that treats classes in a spirit of genuine parity with sets, and we recommend Jean Rubin's lucid exposition of it in her 1967, from which the interested reader will be able to see the points in which it resembles the theory presented here.

Like sets, classes exist when and only when there are many individuals (Theorem 39(iii)). From Separation it follows that a multitude is a class provided it has a subclass (Theorem 43). There are also principles governing classes and various other operations, which follow from their analogues for multitudes and sets as laid out above. The absentee is a pairing principle, since proper pairs are always sets (Theorems 45–48).

All of these principles yield methods to prove that a multitude is a class, but only given that some multitude is a class. Other methods do not rely on such an assumption, starting with the proof that the 'Russell' multitude $\mathbf{x}\cdot\mathbf{x}\notin\mathbf{x}$ is a class, which redeploys the reasoning of Russell's paradox and uses only the logic of quantification. The same goes for the infinite series of descriptions of the same class: $\mathbf{x}\cdot \neg\exists\mathbf{y}(\mathbf{x}\in\mathbf{y} \wedge \mathbf{y}\in\mathbf{x})$, $\mathbf{x}\cdot \neg\exists\mathbf{y}\exists\mathbf{z}(\mathbf{x}\in\mathbf{y} \wedge \mathbf{y}\in\mathbf{z} \wedge \mathbf{z}\in\mathbf{x})$ etc (see Quine 1969, §5). But this is a special case. To prove that other multitudes are classes requires more powerful resources. One route goes via Axiom 1(iii) prohibiting self-membership: if the multitude $\mathbf{x}\cdot A(\mathbf{x})$ is an A provided it is a set, then $\mathbf{x}\cdot A(\mathbf{x})$ must be a class (Theorem 40). This result can be used in swift proofs that $\mathbf{x}\cdot M\mathbf{x}$, $\mathbf{x}\cdot\mathbf{x}=\mathbf{x}$, $\mathbf{x}\cdot S\mathbf{x}$ and $\mathbf{x}\cdot E\mathbf{x}$ are classes. Still more powerful resources are needed for yet other classes. One technique exploits the location of classes outside of the hierarchy of levels (Theorems 41 and 42).

Further necessary and sufficient conditions for classhood are given by Theorem 49, from which it follows that classes are infinitely reproductive, in two directions. Downward: every class has a proper subclass, and so, for any *subclass* of a class, there is a less inclusive subclass of the same class. Upward: for any *subset* of a class, there is a more inclusive subset of the same class. To take the simplest example of the upward phenomenon, adjoining any subset of $\mathbf{x}\cdot\mathbf{x}\notin\mathbf{x}$ to itself gives a more inclusive subset of $\mathbf{x}\cdot\mathbf{x}\notin\mathbf{x}$. This, we contend, is the way to make sense of Russell's notion of a 'self-reproductive process' (1907, p. 36) and Michael Dummett's notion of an 'indefinitely extensible concept' (see his 1991, p. 317 fn. 5, where he acknowledges Russell's priority). A self-reproductive process is embodied by a function which maps any set of elements each having a certain property to another element also having that property but not belonging to the set. In the case of non-self-membership, the function is identity. Russell concluded that 'we can never collect *all* the [items] having the said property into a whole' (1907, p. 36), whereas we say that the property determines a class rather than a set.

§11 Lemmas and theorems.

Here we present our current development of the system. Proofs are given in the Appendix. The proofs and results are to be understood as including the familiar provisos to prevent unintended capture of variables. As before, it is convenient to reserve the variables **u**, **v**, **w** for levels and **h** for histories.

LEMMA 1. *Extensionality for exhaustive description*

(i) $\forall\mathbf{x}(A(\mathbf{x})\leftrightarrow B(\mathbf{x}))\rightarrow\mathbf{x}\cdot A(\mathbf{x})\equiv\mathbf{x}\cdot B(\mathbf{x})$ (ii) $\forall\mathbf{x}(A(\mathbf{x})\leftrightarrow B(\mathbf{x}))\rightarrow\mathbf{x}{:}A(\mathbf{x})\equiv\mathbf{x}{:}B(\mathbf{x})$

Weak identity is needed, since the antecedents do not guarantee that $\mathbf{x}\cdot A(\mathbf{x})$ and $\mathbf{x}{:}A(\mathbf{x})$ exist.

LEMMA 2. *Comprehension*

(i) $m\mathbf{z}(E\mathbf{z}\wedge A(\mathbf{z}))\leftrightarrow \mathsf{E}!(\mathbf{x}\cdot A(\mathbf{x}))$ (iii) $\exists_1\mathbf{z}(E\mathbf{z}\wedge A(\mathbf{z}))\leftrightarrow\mathbf{x}{:}A(\mathbf{x})=\iota\mathbf{x}(E\mathbf{x}\wedge A(\mathbf{x}))$

(ii) $m\mathbf{z}(E\mathbf{z}\wedge A(\mathbf{z}))\leftrightarrow\mathbf{x}{:}A(\mathbf{x})=\mathbf{x}\cdot A(\mathbf{x})$ (iv) $\exists\mathbf{z}(E\mathbf{z}\wedge A(\mathbf{z}))\leftrightarrow\mathsf{E}!(\mathbf{x}{:}A(\mathbf{x}))$

The $\rightarrow$ halves of (i) and (iv) are comprehension principles: $\mathbf{x}\cdot A(\mathbf{x})$ exists if many elements are each A, while $\mathbf{x}{:}A(\mathbf{x})$ exists if at least one element is A. Parts (ii) and (iii) spell out the denotation of $\mathbf{x}{:}A(\mathbf{x})$, according as one or many elements are each A.

LEMMA 3. *What there is*

(i) $\mathsf{E}!a\leftrightarrow Ua\vee Ma$ (ii) $Ma\leftrightarrow Sa\vee Ca$ (iii) $Ea\leftrightarrow m\mathbf{x}U\mathbf{x}\wedge(Ua\vee Sa)$

The $\rightarrow$ halves of (i), (ii), and (iii) divide items into individuals and multitudes, multitudes into sets and classes, and elements into individuals and sets. The $\leftarrow$ half of (i) expresses the strength of the predicates U and M, and together with the $\leftarrow$ half of (ii) and the $\rightarrow$ half of (iii), the strength of S, C and E also follows. The right-hand side of (iii) needs the initial conjunct $m\mathbf{x}U\mathbf{x}$, since if there is but one individual, there is nothing for it to be a member of.

LEMMA 4. *Exhaustive description and reduction*

$a=\mathbf{x}\cdot A(\mathbf{x})\leftrightarrow Ma\wedge\forall\mathbf{y}(\mathbf{y}\in a\leftrightarrow(E\mathbf{y}\wedge A(\mathbf{y})))$

This provides for the introduction or elimination of $\mathbf{x}\cdot A(\mathbf{x})$ via its membership condition.

LEMMA 5. *Membership*

(i) $a\in b\rightarrow Ea\wedge Mb$ (ii) $a=\mathbf{x}\cdot\mathbf{x}\in a\leftrightarrow Ma$ (iii) $a=\mathbf{x}{:}\mathbf{x}\in a\leftrightarrow Ma$

Part (i) says that membership holds only between elements and multitudes. The $\leftarrow$ halves of (ii) and (iii) allow for movement between different expressions for a multitude.

LEMMA 6. *Submultitude*
(i) $a \subseteq b \to Mb$ (ii) $Ma \leftrightarrow a \subseteq a$ (iii) $a \subset b \to \exists \mathbf{x}(\mathbf{x} \notin a \wedge \mathbf{x} \in b)$
Elementary properties of $\subseteq$ and $\subset$

LEMMA 7. *Zilch*
(i) $\neg \mathsf{E}!O$ (iv) $\neg EO$ (vii) $\neg VO$
(ii) $\neg UO \wedge \neg MO$ (v) $a \notin O \wedge O \notin a$ (viii) HO
(iii) $\neg SO \wedge \neg CO$ (vi) $a \not\subseteq O \wedge O \not\subseteq a$ (ix) $\neg \mathsf{E}!a \leftrightarrow a \equiv O$
Parts (i)-(vii) express the strength of leading predicates. Part (viii) is the exception: H is weak, since zilch is a history. Part (ix) says that a is non-existent iff a is (weakly identical to) zilch.

LEMMA 8. *Levels* *Let Va, then* (i) Ma, (ii) Ea, (iii) Sa, (iv) $\forall \mathbf{y}(U\mathbf{y} \to \mathbf{y} \in a)$.
Levels are multitudes, elements, and sets, and have every individual as a member.

LEMMA 9. *The first two levels*
(i) $V_1 \equiv \mathbf{z}{\cdot}U\mathbf{z} \equiv \mathrm{acc}(O) \equiv \mathrm{acc}(V_1)$ (v) $\mathsf{E}!V_2 \leftrightarrow MV_2 \wedge \forall \mathbf{y}(\mathbf{y} \in V_2 \leftrightarrow (U\mathbf{y} \vee \mathbf{y} \subseteq V_1))$
(ii) $\mathsf{E}!V_1 \leftrightarrow MV_1 \wedge \forall \mathbf{y}(\mathbf{y} \in V_1 \leftrightarrow U\mathbf{y})$ (vi) $\mathsf{E}!V_2 \leftrightarrow \mathsf{E}!V_1$
(iii) $\exists \mathbf{x} M\mathbf{x} \leftrightarrow V_1 = \mathbf{z}{\cdot}U\mathbf{z}$ (vii) $\mathsf{E}!V_2 \leftrightarrow V_1 \in V_2$
(iv) $V_2 \equiv \mathbf{z}{\cdot}(U\mathbf{z} \vee \mathbf{z} \subseteq V_1)$ (viii) $V_1 \neq V_2$
A recurrent feature of the theorems and their proofs is the separate treatment of the first two levels. We prepare the ground for this by including their main peculiarities here. In particular, provided there is at least one multitude, V_1 is the multitude of all the individuals, and V_2 is the multitude of all the individuals plus all the submultitudes of V_1, whence $V_1 \in V_2$ and $V_1 \neq V_2$.

THEOREM 1. *Separation*
(i) $S\mathbf{x} \wedge m\mathbf{y}(\mathbf{y} \in \mathbf{x} \wedge A(\mathbf{y})) \to S(\mathbf{z}{\cdot}\mathbf{z} \in \mathbf{x} \wedge A(\mathbf{z}))$
(ii) $S\mathbf{x} \leftrightarrow \exists \mathbf{u}\ \mathbf{x} \subseteq \mathbf{u}$
(iii) $S(\mathbf{x}{\cdot}A(\mathbf{x})) \leftrightarrow (m\mathbf{y}(E\mathbf{y} \wedge A(\mathbf{y})) \wedge \exists \mathbf{u} \forall \mathbf{z}((E\mathbf{z} \wedge A(\mathbf{z})) \to \mathbf{z} \in \mathbf{u}))$
The scheme (i) is derived from Axiom 2(iii). (ii) gives a necessary and sufficient condition for $\mathbf{x}$ to be a set, namely that $\mathbf{x}$ is a submultitude of some level. From (ii) we derive the scheme (iii). The $\leftarrow$ halves of (ii) and (iii) are useful principles of separation from levels.

THEOREM 2. *Intersection*
(i) $\exists \mathbf{z}(\mathbf{z} \in \mathbf{x} \wedge \mathbf{z} \in \mathbf{y}) \leftrightarrow \mathsf{E}!(\mathbf{x} \cap \mathbf{y})$
(ii) *Let* $\exists_1 \mathbf{z}(\mathbf{z} \in \mathbf{x} \wedge \mathbf{z} \in \mathbf{y})$, *then* $E(\mathbf{x} \cap \mathbf{y})$.
(iii) *Let* $m\mathbf{z}(\mathbf{z} \in \mathbf{x} \wedge \mathbf{z} \in \mathbf{y})$, *then* $M(\mathbf{x} \cap \mathbf{y})$.
(iv) *Let* $m\mathbf{z}(\mathbf{z} \in \mathbf{x} \wedge \mathbf{z} \in \mathbf{y})$ *and* $(S\mathbf{x} \vee S\mathbf{y})$, *then* $S(\mathbf{x} \cap \mathbf{y})$.
Part (i) spells out the existence conditions of the intersection $\mathbf{x} \cap \mathbf{y}$, while (ii)-(iv) characterise it, given assumptions about the number and nature of the common members.

THEOREM 3. *Histories I* *Let* $\mathbf{v} = \mathrm{acc}(\mathbf{h})$, *then* (i) $\mathbf{x} \in \mathbf{h} \to \mathbf{x} \in \mathbf{v}$, (ii) $\mathbf{v} \neq V_1 \to M\mathbf{h} \wedge \mathbf{h} \neq V_1$.
THEOREM 4. *Histories II* *Let* $\mathbf{h} \neq V_1$ *and* $\mathbf{x} \in \mathbf{h}$, *then* $M\mathbf{x}$.
THEOREM 5. *Histories III* *Let* $\mathbf{h} \neq V_1$, $\mathbf{x} \in \mathbf{h}$, $\mathbf{x} \neq V_1$ *and* $\mathbf{x} \neq V_2$. *Then* (i) $\mathbf{x} = \mathrm{acc}(\mathbf{h} \cap \mathbf{x})$, (ii) $\mathsf{E}!(\mathbf{h} \cap \mathbf{x})$, (iii) $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z} \in \mathbf{h} \wedge \mathbf{z} \in \mathbf{x})$, (iv) $S(\mathbf{h} \cap \mathbf{x})$.
THEOREM 6. *Histories IV* *Let* $\mathbf{h} \neq V_1$ *and* $\mathbf{x} \in \mathbf{h}$, *then* $H(\mathbf{h} \cap \mathbf{x})$ *and* $V\mathbf{x}$.

Theorems 3-6 state basic properties of histories and their members. Theorem 5(iii) plays a critical role in the proofs of subsequent theorems, where we need to infer $\mathbf{h}\cap\mathbf{x} = \mathbf{z}\cdot(\mathbf{z}\in\mathbf{h} \wedge \mathbf{z}\in\mathbf{x})$ and so $\mathbf{y}\in\mathbf{h}\cap\mathbf{x} \leftrightarrow \mathbf{y}\in\mathbf{h} \wedge \mathbf{y}\in\mathbf{x}$. Caution is needed with this inference, since if there is but one common member of **h** and **x**, the intersection $\mathbf{h}\cap\mathbf{x}$ is that member itself. This is the one place where the absence of singleton multitudes presents a serious challenge. For the inferences to go through, we need to have shown that there are many common members of **h** and **x**. Theorem 5(iii) does this, subject to the conditions in the hypothesis. The conjunct $E\mathbf{z}$ in $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\in\mathbf{h} \wedge \mathbf{z}\in\mathbf{x})$ is strictly redundant but we include it to allow for direct application of lemma 2(ii).

THEOREM 7. *$\in$ is well-founded on any history* *Let* $\mathbf{x}\subseteq\mathbf{h}$, *then* $\exists\mathbf{y}(\mathbf{y}\in\mathbf{x} \wedge \mathbf{x}\cap\mathbf{y}\equiv O)$.
In English: any submultitude of a history has a member whose intersection with that submultitude is zilch. This is one of three foundation principles of increasing generality; see Theorems 13 and 18 for the others.

THEOREM 8. *Levels are transitive sets* *Let* $\mathbf{x}\in\mathbf{y}$ *and* $\mathbf{y}\in\mathbf{v}$, *then* $\mathbf{x}\in\mathbf{v}$.
COROLLARY. *Let* $M\mathbf{y}$ *and* $\mathbf{y}\in\mathbf{v}$, *then* $\mathbf{y}\subseteq\mathbf{v}$.
In particular, membership between levels is transitive. It also follows that if a set is a member of a level, it is a member of all higher levels. The corollary tells us that it is a subset of all those levels too, the original included.

THEOREM 9. *Levels are hereditary sets* *Let* $\mathbf{x}\subseteq\mathbf{y}$ *and* $\mathbf{y}\in\mathbf{v}$, *then* $\mathbf{x}\in\mathbf{v}$.
In particular, if a set is a subset of a level, it is a member of all higher levels.

THEOREM 10. *Lower levels I*
(i) $\neg\exists\mathbf{w}\ \mathbf{w}\in V_1$ *and* $\mathbf{w}{:}\mathbf{w}\in V_1 \equiv O$
(ii) *Let* $\mathsf{E}!V_2$, *then* $\exists_1\mathbf{w}\ \mathbf{w}\in V_2$ *and* $\mathbf{w}{:}\mathbf{w}\in V_2 = \iota\mathbf{w}(\mathbf{w}\in V_2) = V_1$.
(iii) *Let* $\mathsf{E}!\mathbf{v}$ *and* $\mathbf{v}\neq V_1$ *and* $\mathbf{v}\neq V_2$, *then* $m\mathbf{w}\ \mathbf{w}\in\mathbf{v}$ *and* $\mathbf{w}{:}\mathbf{w}\in\mathbf{v} = \mathbf{w}\cdot\mathbf{w}\in\mathbf{v}$.
These cover the three possibilities for the number of levels lower than a given level: none, one or many.

THEOREM 11. *Lower levels II*
(i) *Let* $\mathbf{v}=V_1$ *or* $\mathbf{v}=V_2$, *then* $\mathbf{v}=\mathrm{accum}(\mathbf{w}{:}\mathbf{w}\in\mathbf{v})$.
(ii) *Let* $\mathsf{E}!\mathbf{v}$ *and* $\mathbf{v}\neq V_1$ *and* $\mathbf{v}\neq V_2$, *then* $\mathbf{v}=\mathrm{acc}(\mathbf{w}{:}\mathbf{w}\in\mathbf{v})$.

THEOREM 12. *Lower levels III* $H(\mathbf{w}{:}\mathbf{w}\in\mathbf{v})$
Theorems 11 and 12 together ensure that any level **v** has $\mathbf{w}{:}\mathbf{w}\in\mathbf{v}$ as a history.

THEOREM 13. *Foundation for levels*
(i) *Let* $\exists\mathbf{u}\ \mathbf{u}\in\mathbf{x}$, *then* $\exists\mathbf{v}(\mathbf{v}\in\mathbf{x} \wedge \neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{v} \wedge \mathbf{w}\in\mathbf{x}))$.
(ii) *Let* $\exists\mathbf{u}\ A(\mathbf{u})$, *then* $\exists\mathbf{v}(A(\mathbf{v}) \wedge \neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{v} \wedge A(\mathbf{w})))$.
Part (i) says that there is a lowest level belonging to a multitude provided some level belongs to it. (ii) is the corresponding scheme. It follows that membership between levels is well-founded: any submultitude of the multitude of all levels has a lowest member.

THEOREM 14. *Comparability of levels* $\mathbf{v}\in\mathbf{w} \vee \mathbf{v}=\mathbf{w} \vee \mathbf{w}\in\mathbf{v}$
In other words, given any two levels, one is lower than the other. Theorems 13 and 14 together entail the uniqueness of lowest levels, as stated in the following theorem.

THEOREM 15. *The lowest level principle*

(i) *Let* $\exists\mathbf{u}\ \mathbf{u}\in\mathbf{x}$, *then* $\exists_1\mathbf{v}(\mathbf{v}\in\mathbf{x} \wedge \neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{v} \wedge \mathbf{w}\in\mathbf{x}))$.

(ii) *Let* $\exists\mathbf{u}\ A(\mathbf{u})$, *then* $\exists_1\mathbf{v}(A(\mathbf{v}) \wedge \neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{v} \wedge A(\mathbf{w})))$.

THEOREM 16. *Uniqueness of histories*

(i) *Let* V_1=accum($\mathbf{h}$), *then* $\mathbf{h} \equiv \mathbf{w}{:}\mathbf{w}\in V_1$.

(ii) *Let* V_2=accum($\mathbf{h}$), *then* $\mathbf{h} = \mathbf{w}{:}\mathbf{w}\in V_2$.

(iii) *Let* $\mathbf{v}\neq V_1$ and $\mathbf{v}\neq V_2$ *and* $\mathbf{v}$=acc($\mathbf{h}$), *then* $\mathbf{h} = \mathbf{w}{:}\mathbf{w}\in\mathbf{v}$.

This theorem ensures that a level $\mathbf{v}$ has $\mathbf{w}{:}\mathbf{w}\in\mathbf{v}$ as its unique history.

THEOREM 17. *Sets and levels I*

(i) $S\mathbf{x}\leftrightarrow\mathsf{E}!V^*(\mathbf{x})$

(ii) $\mathbf{x}\notin V^*(\mathbf{x})$

(iii) $V^*(\mathbf{u})=\mathbf{u}$

(iv) *Let* $S\mathbf{x} \wedge S\mathbf{y} \wedge \mathbf{x}\in\mathbf{y}$, *then* $V^*(\mathbf{x})\in V^*(\mathbf{y})$.

(v) *Let* $S\mathbf{x} \wedge S\mathbf{y} \wedge \mathbf{x}\subseteq\mathbf{y}$, *then* $V^*(\mathbf{x})\subseteq V^*(\mathbf{y})$.

(vi) $S\mathbf{x}\leftrightarrow(M\mathbf{x} \wedge \exists\mathbf{u}\forall\mathbf{y}((S\mathbf{y} \wedge \mathbf{y}\in\mathbf{x})\rightarrow V^*(\mathbf{y})\in\mathbf{u}))$

Parts (i)-(v) state properties of the V^* function. Part (vi) gives a necessary and sufficient condition for a multitude to be a set, namely that there is some level higher than all the levels of the sets, if any, among its members.

THEOREM 18. *Foundation for multitudes* *Let* $M\mathbf{x}$, *then* $\exists\mathbf{y}(\mathbf{y}\in\mathbf{x} \wedge \mathbf{x}\cap\mathbf{y}\equiv O)$.

In English: any multitude has a member whose intersection with that multitude is zilch.

THEOREM 19. *Generalized intersection*

(i) $\exists\mathbf{x}(\forall\mathbf{y}(A(\mathbf{y})\rightarrow\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z})) \leftrightarrow \mathsf{E}!\cap\mathbf{x}A(\mathbf{x})$

(ii) *Let* $\exists_1\mathbf{x}(\forall\mathbf{y}(A(\mathbf{y})\rightarrow\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z}))$, *then* $E(\cap\mathbf{x}A(\mathbf{x}))$.

(iii) *Let* $m\mathbf{x}(\forall\mathbf{y}(A(\mathbf{y})\rightarrow\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z}))$, *then* $M(\cap\mathbf{x}A(\mathbf{x}))$.

(iv) *Let* $m\mathbf{x}(\forall\mathbf{y}(A(\mathbf{y})\rightarrow\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge S\mathbf{z}))$, *then* $S(\cap\mathbf{x}A(\mathbf{x}))$.

COROLLARY. *Let* $m\mathbf{z}\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow\mathbf{z}\in\mathbf{y}) \wedge \exists\mathbf{z}\ \mathbf{z}\in\mathbf{x}$, *then* $S\cap\mathbf{x}$.

This theorem does for generalized intersection what Theorem 2 does for the binary version.

THEOREM 20. *Union* (i) $M\mathbf{x} \wedge M\mathbf{y} \leftrightarrow M(\mathbf{x}\cup\mathbf{y})$ (ii) $S\mathbf{x} \wedge S\mathbf{y} \leftrightarrow S(\mathbf{x}\cup\mathbf{y})$

The $\rightarrow$ half of part (i) says that if $\mathbf{x}$ and $\mathbf{y}$ are multitudes, so too is the union $\mathbf{x}\cup\mathbf{y}$. It is easily proved via comprehension. The $\rightarrow$ half of (ii) says more specifically that the union is a set provided that $\mathbf{x}$ and $\mathbf{y}$ are sets. Its proof uses separation from levels (see Theorem 1). Similar remarks apply to the following theorem.

THEOREM 21. *Generalized union*

(i) $(\forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z})) \leftrightarrow M(\cup\mathbf{x}A(\mathbf{x}))$

(ii) $(\forall\mathbf{y}(A(\mathbf{y})\rightarrow S\mathbf{y}) \wedge S(\mathbf{z}{:}A(\mathbf{z}))) \leftrightarrow S(\cup\mathbf{x}A(\mathbf{x}))$

COROLLARIES. (i) $(\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow S\mathbf{y}) \wedge M\mathbf{x}) \leftrightarrow M\cup\mathbf{x}$

(ii) $(\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow S\mathbf{y}) \wedge S\mathbf{x}) \leftrightarrow S\cup\mathbf{x}$

THEOREM 22. *Sets and levels II* (i) $S\mathbf{x}\leftrightarrow(M\mathbf{x} \wedge \exists\mathbf{u}\ \mathbf{x}\in\mathbf{u})$ (ii) $S\mathbf{x}\leftrightarrow(M\mathbf{x} \wedge \mathsf{E}!V^{\dagger}(\mathbf{x}))$

Part (i) means that a multitude is a set iff it is a member of some level. By Theorem 15 there is a unique lowest level of which it is member, which is expressed in (ii) using $V^{\dagger}$.

THEOREM 23. *Pairing* (i) $(E\mathbf{x} \wedge E\mathbf{y} \wedge \mathbf{x}\neq\mathbf{y})\leftrightarrow\mathsf{E}!|\mathbf{x}, \mathbf{y}|$ (ii) $\mathsf{E}!|\mathbf{x}, \mathbf{y}|\leftrightarrow S|\mathbf{x}, \mathbf{y}|$

The proper pair $|\mathbf{x}, \mathbf{y}|$ exists iff (i) $\mathbf{x}$ and $\mathbf{y}$ are distinct elements, iff (ii) it is a set.

THEOREM 24. *Adjunction* (i) $M\mathbf{x} \wedge E\mathbf{y} \leftrightarrow M(\mathbf{x}\oplus\mathbf{y})$ (ii) $S\mathbf{x} \wedge E\mathbf{y} \leftrightarrow S(\mathbf{x}\oplus\mathbf{y})$

The $\rightarrow$ half of part (i) says that the adjunction $\mathbf{x}\oplus\mathbf{y}$ is a multitude provided $\mathbf{x}$ is a multitude and $\mathbf{y}$ an element. This is proved by comprehension, whereas the proof of the $\rightarrow$ half of (ii) uses separation from levels to show that $\mathbf{x}\oplus\mathbf{y}$ is a set provided $\mathbf{x}$ is a set and $\mathbf{y}$ an element. Similar remarks apply to the proof strategies used to show that power multitudes and power-plus multitudes are multitudes or sets, respectively (see the next two theorems).

THEOREM 25. *Power multitude*

- (i) *Let* $\exists\mathbf{y}_1\exists\mathbf{y}_2(\mathbf{y}_1\in\mathbf{x} \wedge \mathbf{y}_2\in\mathbf{x} \wedge \mathbf{y}_1\neq\mathbf{y}_2 \wedge \forall\mathbf{y}_3(\mathbf{y}_3\in\mathbf{x}\rightarrow (\mathbf{y}_3=\mathbf{y}_1 \vee \mathbf{y}_3=\mathbf{y}_2)))$, *then* $\imath\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\subseteq\mathbf{x})=\mathbf{x} \wedge P(\mathbf{x})=\mathbf{x}$.
- (ii) *Let* $\exists\mathbf{y}_1\exists\mathbf{y}_2\exists\mathbf{y}_3(\mathbf{y}_1\in\mathbf{x} \wedge \mathbf{y}_2\in\mathbf{x} \wedge \mathbf{y}_3\in\mathbf{x} \wedge \mathbf{y}_1\neq\mathbf{y}_2 \wedge \mathbf{y}_1\neq\mathbf{y}_3 \wedge \mathbf{y}_2\neq\mathbf{y}_3)$, *then* $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\subseteq\mathbf{x}) \wedge P(\mathbf{x})=\mathbf{y}{\cdot}\mathbf{y}\subseteq\mathbf{x}$.
- (iii) $M\mathbf{x} \leftrightarrow M(P(\mathbf{x}))$
- (iv) $S\mathbf{x} \leftrightarrow S(P(\mathbf{x}))$

In general, the power multitude of a multitude comprises its many subsets. Parts (i) and (ii) jointly entail that pair sets are the only exceptional case, since they are the sole subsets of themselves, and so are their own power multitudes. Part (iii) says that $P(\mathbf{x})$ is a multitude iff $\mathbf{x}$ is too, while (iv) says that $P(\mathbf{x})$ is a set iff $\mathbf{x}$ is too.

THEOREM 26. *Power-plus multitude* (i) $M\mathbf{x} \leftrightarrow M(P^+(\mathbf{x}))$ (ii) $S\mathbf{x} \leftrightarrow S(P^+(\mathbf{x}))$

Part (i) says that $P^+(\mathbf{x})$ is a multitude iff $\mathbf{x}$ is too, while (ii) says that $P^+(\mathbf{x})$ is a set iff $\mathbf{x}$ is too.

THEOREM 27. *Membership and proper submultitude among levels* $\mathbf{v}\in\mathbf{w}\leftrightarrow\mathbf{v}\subset\mathbf{w}$

Since membership well-orders levels, and this theorem says that membership and proper submultitude are equivalent among levels, so proper submultitude well-orders levels too.

THEOREM 28. *Numbers of individuals*

(i) $m\mathbf{x}U\mathbf{x}\leftrightarrow\exists\mathbf{x}M\mathbf{x}$	(iv) $m\mathbf{x}U\mathbf{x}\leftrightarrow m\mathbf{x}\ \mathbf{x}=\mathbf{x}$
(ii) $m\mathbf{x}U\mathbf{x}\leftrightarrow\exists\mathbf{x}V\mathbf{x}$	(v) $\exists_1\mathbf{x}U\mathbf{x}\leftrightarrow\exists_1\mathbf{x}\ \mathbf{x}=\mathbf{x}$
(iii) $m\mathbf{x}U\mathbf{x}\leftrightarrow\exists\mathbf{x}S\mathbf{x}$	(vi) $\neg\exists\mathbf{x}U\mathbf{x}\leftrightarrow\neg\exists\mathbf{x}\ \mathbf{x}=\mathbf{x}$

The existence of many individuals (Axiom of Plurality) is necessary and sufficient for the existence of (i) multitudes, (ii) levels, (iii) sets. Parts (iv)-(vi) correlate the number of individuals with the number of items: there are many/one/no individuals iff there are many/one/no items.

THEOREM 29. *The lowest level* (i) $m\mathbf{x}U\mathbf{x}\leftrightarrow \mathsf{E}!V_1$ (ii) $\exists\mathbf{x}V\mathbf{x}\leftrightarrow\mathsf{E}!V_1$ (iii) $V_1 \equiv \imath\mathbf{v}(\neg\exists\mathbf{w}\ \mathbf{w}\in\mathbf{v})$

V_1 exists iff (i) there are many individuals, iff (ii) there is any level at all. Part (iii) identifies it as the lowest level.

THEOREM 30. *Levels next above I* $\mathsf{E}!\mathbf{u} \leftrightarrow \mathsf{E}!\mathbf{u}'$

This says that a level exists iff the level next above exists.

THEOREM 31. *Levels next above II* (i) $\mathbf{u}' = \mathbf{x}{\cdot}(U\mathbf{x} \vee \mathbf{x}\subseteq\mathbf{u})$ (ii) $\mathbf{u}' = P^+(\mathbf{u})$

This identifies the level next above a given level $\mathbf{u}$ with (i) the multitude comprising all the individuals plus all the submultitudes of $\mathbf{u}$, and with (ii) the power-plus multitude of $\mathbf{u}$, that is to say, the multitude comprising all the members and submultitudes of $\mathbf{u}$.

THEOREM 32. *Sets and levels III* *Let* $S\mathbf{x}$, *then* $(V^*(\mathbf{x}))' = V^\dagger(\mathbf{x})$.
This means that the level next above the level of a set is the lowest level of which the set is a member.

THEOREM 33. *Levels next above III*
(i) *Let* $\exists\mathbf{y}_1\exists\mathbf{y}_2(U\mathbf{y}_1 \wedge U\mathbf{y}_2 \wedge \mathbf{y}_1{\neq}\mathbf{y}_2 \wedge \forall\mathbf{y}_3(U\mathbf{y}_3\rightarrow (\mathbf{y}_3{=}\mathbf{y}_1 \vee \mathbf{y}_3{=}\mathbf{y}_2)))$, *then* $\iota\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}{\subseteq}V_1){=}V_1 \wedge P(V_1){=}V_1 \wedge \forall\mathbf{u}(\mathbf{u}{\neq}V_1\rightarrow(m\mathbf{z}\ \mathbf{z}{\subseteq}\mathbf{u} \wedge P(\mathbf{u}){=}\mathbf{y}{\cdot}\mathbf{y}{\subseteq}\mathbf{u}))$.
(ii) *Let* $\exists\mathbf{y}_1\exists\mathbf{y}_2\exists\mathbf{y}_3(U\mathbf{y}_1 \wedge U\mathbf{y}_2 \wedge U\mathbf{y}_3 \wedge \mathbf{y}_1{\neq}\mathbf{y}_2 \wedge \mathbf{y}_1{\neq}\mathbf{y}_3 \wedge \mathbf{y}_2{\neq}\mathbf{y}_3)$, *then* $\forall\mathbf{u}(m\mathbf{z}\ \mathbf{z}{\subseteq}\mathbf{u} \wedge P(\mathbf{u}){=}\mathbf{y}{\cdot}\mathbf{y}{\subseteq}\mathbf{u})$.
(iii) $\exists_1\mathbf{z}\ \mathbf{z}{\subseteq}\mathbf{u} \leftrightarrow \mathbf{u}'{=}V_1{\oplus}P(\mathbf{u})$
(iv) $m\mathbf{z}\ \mathbf{z}{\subseteq}\mathbf{u} \leftrightarrow \mathbf{u}'{=}V_1{\cup}P(\mathbf{u})$

Part (iv) means that the level next above a level $\mathbf{u}$ is the union of V_1 with the power multitude of $\mathbf{u}$, provided $\mathbf{u}$ has many subsets. Parts (i) and (ii) entail that in general a level has many subsets, with part (i) identifying the single exception, namely V_1 in the special case when there are exactly two individuals, in which case V_1 is both the only subset of itself and the power multitude of itself, and so by parts (iii) and (iv), the level next above V_1 is the adjunction $V_1{\oplus}P(V_1)$, not the union $V_1{\cup}P(V_1)$.

THEOREM 34. *Limit levels* (i) $m\mathbf{x}U\mathbf{x} \leftrightarrow \exists\mathbf{x}L\mathbf{x}$ (ii) $L\mathbf{u} \leftrightarrow \mathbf{u} = \cup\mathbf{v}\ \mathbf{v}{\in}\mathbf{u}$
Part (i) says that a limit level exists iff there are many individuals. (ii) says that a level is a limit level iff it is the union of its lower levels.

The next three theorems are designed to yield a representation $\mathbf{N}^*$ of the set of natural numbers as the set of finite levels, given the Axiom of Plurality. We first define V_ω to be the lowest limit level. In symbols, $V_\omega =_{df} \iota\mathbf{x}(L\mathbf{x} \wedge \neg\exists\mathbf{y}(\mathbf{y}{\in}\mathbf{x} \wedge L\mathbf{y}))$. Then we define a to be *inductive* if (i) V_1 belongs to a and (ii) the level next above any level belonging to a also belongs to a. In symbols, $Ia =_{df} V_1{\in}a \wedge \forall\mathbf{x}((V\mathbf{x} \wedge \mathbf{x}{\in}a)\rightarrow\mathbf{x}'{\in}a)$. Finally we define $\mathbf{N}^*$ as the intersection $\cap\mathbf{x}I\mathbf{x}$.

THEOREM 35. *The lowest limit level* $\mathsf{E}!V_\omega \leftrightarrow \exists\mathbf{x}L\mathbf{x}$

THEOREM 36. *V_ω is inductive* *Let* $\mathsf{E}!V_\omega$, *then* $I(V_\omega)$.

THEOREM 37. *$\mathbf{N}^*$ is an inductive set* *Let* $m\mathbf{x}U\mathbf{x}$, *then* $S(\mathbf{N}^*) \wedge I(\mathbf{N}^*)$.

We define the ordered pair $<a, b>$ as $||\,|[a, V_1], [a, V_2]|, V_1|, |[b, V_1], [b, V_2]|, V_2||$.

THEOREM 38. *Ordered pairs*
Let $m\mathbf{z}U\mathbf{z} \wedge \neg C\mathbf{x}_1 \wedge \neg C\mathbf{x}_2 \wedge \neg C\mathbf{y}_1 \wedge \neg C\mathbf{y}_2$. *Then* (i) $E!<\mathbf{x}_1, \mathbf{x}_2>$ *and* (ii) $<\mathbf{x}_1, \mathbf{x}_2> = <\mathbf{y}_1, \mathbf{y}_2> \leftrightarrow (\mathbf{x}_1{\equiv}\mathbf{y}_1 \wedge \mathbf{x}_2{\equiv}\mathbf{y}_2)$.
Part (i) states the existence of ordered pairs and (ii) their so-called characteristic property. Besides the Axiom of Plurality, the hypothesis needs to say that the putative coordinates are not classes, while allowing them to be anything else: zilch or an individual or a set.

THEOREM 39. *Existence of classes*
(i) *Let* $m\mathbf{x}U\mathbf{x}$, *then* $\mathsf{E}!(\mathbf{x}{\cdot}\mathbf{x}{\notin}\mathbf{x})$.
(ii) *Let* $\mathsf{E}!(\mathbf{x}{\cdot}\mathbf{x}{\notin}\mathbf{x})$, *then* $C(\mathbf{x}{\cdot}\mathbf{x}{\notin}\mathbf{x})$.
(iii) $m\mathbf{x}U\mathbf{x} \leftrightarrow \exists\mathbf{x}C\mathbf{x}$

Part (i) means that the 'Russell' multitude exists provided there are many individuals, and (ii) means that it is a class. Part (iii) follows: classes exist just in case there are many individuals.

THEOREM 40. *Classes and non-self-membership*

Let $\mathsf{E}!(\mathbf{x}\cdot A(\mathbf{x})) \wedge S(\mathbf{x}\cdot A(\mathbf{x}))\rightarrow A(\mathbf{x}\cdot A(\mathbf{x}))$, *then* $C(\mathbf{x}\cdot A(\mathbf{x}))$.

This gives a sufficient condition for a multitude $\mathbf{x}\cdot A(\mathbf{x})$ to be a class, namely that it is an A provided it is a set.

THEOREM 41. *Classes and levels*

(i) $C\mathbf{x}\leftrightarrow(M\mathbf{x} \wedge \neg\exists\mathbf{u}\ \mathbf{x}\subseteq\mathbf{u})$

(ii) $C\mathbf{x}\leftrightarrow(M\mathbf{x} \wedge \forall\mathbf{u}\exists\mathbf{y}(\mathbf{y}\in\mathbf{x} \wedge \mathbf{u}\in V^*(\mathbf{y}))$

(iii) $C\mathbf{x}\leftrightarrow(M\mathbf{x} \wedge \neg\exists\mathbf{u}\ \mathbf{x}\in\mathbf{u})$

(iv) *Let* $M\mathbf{x} \wedge \forall\mathbf{u}\exists\mathbf{y}(\mathbf{y}\in\mathbf{x} \wedge (\mathbf{u}\in\mathbf{y} \vee \mathbf{u}\subseteq\mathbf{y}))$, *then* $C\mathbf{x}$.

Parts (i)-(iii) each give a necessary and sufficient condition for a multitude to be a class, namely (i) that it is not a submultitude of any level, (ii) that for every level **u** there is some member of the multitude whose level is higher than **u**, (iii) that it is not a member of any level. Part (iv) gives a way for a multitude to fail to be a member of any level, and so provides a general method to show that a wide range of multitudes are classes. A sample is presented in the following theorem.

THEOREM 42. *Classes: an illustrative sample*

(i) *Let* $\mathsf{E}!(\mathbf{x}\cdot\mathbf{y}\in\mathbf{x})$, *then* $C(\mathbf{x}\cdot\mathbf{y}\in\mathbf{x})$.

(ii) *Let* $\mathsf{E}!(\mathbf{x}\cdot\mathbf{y}\notin\mathbf{x})$, *then* $C(\mathbf{x}\cdot\mathbf{y}\notin\mathbf{x})$.

(iii) *Let* $\mathsf{E}!(\mathbf{x}\cdot\mathbf{x}\notin\mathbf{y}) \wedge \neg C\mathbf{y}$, *then* $C(\mathbf{x}\cdot\mathbf{x}\notin\mathbf{y})$.

(iv) *Let* $\mathsf{E}!(\mathbf{x}\cdot\mathbf{x}\neq\mathbf{y})$, *then* $C(\mathbf{x}\cdot\mathbf{x}\neq\mathbf{y})$.

(v) *Let* $\mathsf{E}!(\mathbf{x}\cdot\mathbf{y}\subseteq\mathbf{x})$, *then* $C(\mathbf{x}\cdot\mathbf{y}\subseteq\mathbf{x})$.

(vi) *Let* $\mathsf{E}!(\mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=P(\mathbf{y}))$, *then* $C(\mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=P(\mathbf{y}))$.

(vii) *Let* $\mathsf{E}!(\mathbf{x}\cdot\exists\mathbf{y}\exists\mathbf{z}\ \mathbf{x}=|\mathbf{y}, \mathbf{z}|)$, *then* $C(\mathbf{x}\cdot\exists\mathbf{y}\exists\mathbf{z}\ \mathbf{x}=|\mathbf{y}, \mathbf{z}|)$.

(viii) *Let* $\mathsf{E}!(\mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=\cup\mathbf{y})$, *then* $C(\mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=\cup\mathbf{y})$.

(ix) *Let* $\mathsf{E}!(\mathbf{x}\cdot H\mathbf{x})$, *then* $C(\mathbf{x}\cdot H\mathbf{x})$.

(x) *Let* $\mathsf{E}!(\mathbf{x}\cdot V\mathbf{x})$, *then* $C(\mathbf{x}\cdot V\mathbf{x})$.

Further techniques for proving that a multitude is (or is not) a class are provided by theorems 43–48 below, which govern classes and various operations.

THEOREM 43. *Classes and separation*

(i) *Let* $C\mathbf{x} \wedge \mathbf{x}\subseteq\mathbf{y}$, *then* $C\mathbf{y}$.

(ii) *Let* $C(\mathbf{z}\cdot\ \mathbf{z}\in\mathbf{x} \wedge A(\mathbf{z}))$, *then* $C\mathbf{x}$.

THEOREM 44. *Classes and intersection*

(i) *Let* $C(\mathbf{x}\cap\mathbf{y})$, *then* $C\mathbf{x} \wedge C\mathbf{y}$.

(ii) *Let* $C(\cap\mathbf{x}A(\mathbf{x}))$, *then* $\forall\mathbf{z}(A(\mathbf{z})\rightarrow C\mathbf{z})$.

THEOREM 45. *Classes and union*

(i) $(M\mathbf{x} \wedge M\mathbf{y} \wedge (C\mathbf{x} \vee C\mathbf{y})) \leftrightarrow C(\mathbf{x}\cup\mathbf{y})$

(ii) $(\forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge (\exists\mathbf{z}(A(\mathbf{z}) \wedge C\mathbf{z}) \vee C(\mathbf{z}{:}A(\mathbf{z})))) \leftrightarrow C(\cup\mathbf{x}A(\mathbf{x}))$

THEOREM 46. *Classes and pairing*

(i) *Let* $\mathsf{E}!\mathbf{x} \wedge \mathsf{E}!\mathbf{y} \wedge \mathbf{x}\neq\mathbf{y}$, *then* $\neg\mathsf{E}!|\mathbf{x}, \mathbf{y}| \leftrightarrow C\mathbf{x} \vee C\mathbf{y}$.

(ii) $\neg\exists\mathbf{x}\exists\mathbf{y}C|\mathbf{x}, \mathbf{y}|$

THEOREM 47. *Classes and adjunction* $C\mathbf{x} \wedge E\mathbf{y} \leftrightarrow C(\mathbf{x}\oplus\mathbf{y})$

THEOREM 48. *Classes, power and power-plus*
(i) $C\mathbf{x}\leftrightarrow C(P(\mathbf{x}))$ (ii) $C\mathbf{x} \leftrightarrow C(P^{+}(\mathbf{x}))$

THEOREM 49. *Classes and reproductivity*
(i) $C\mathbf{x}\leftrightarrow\exists\mathbf{y}(C\mathbf{y} \wedge \mathbf{y}\subset\mathbf{x})$
(ii) $C\mathbf{x}\leftrightarrow(M\mathbf{x} \wedge \forall\mathbf{y}((S\mathbf{y} \wedge \mathbf{y}\subseteq\mathbf{x}) \rightarrow \exists\mathbf{z}(S\mathbf{z} \wedge \mathbf{y}\subset\mathbf{z} \wedge \mathbf{z}\subset\mathbf{x})))$
This gives two further necessary and sufficient conditions for a multitude to be a class, namely (i) that it has a proper subclass, and (ii) for any subset of it, there is a more inclusive proper subset of it.

Appendix. Proofs

LEMMA 1. *Extensionality for exhaustive description*

(i) $\forall\mathbf{x}(A(\mathbf{x})\leftrightarrow B(\mathbf{x})) \rightarrow \mathbf{x}{\cdot}A(\mathbf{x}) \equiv \mathbf{x}{\cdot}B(\mathbf{x})$

(ii) $\forall\mathbf{x}(A(\mathbf{x})\leftrightarrow B(\mathbf{x})) \rightarrow \mathbf{x}{:}A(\mathbf{x}) \equiv \mathbf{x}{:}B(\mathbf{x})$

PROOF OF (i). Suppose $\forall\mathbf{x}(A(\mathbf{x})\leftrightarrow B(\mathbf{x}))$. Then $\iota\mathbf{y}(M\mathbf{y} \wedge \forall\mathbf{x}(\mathbf{x}\in\mathbf{y}\leftrightarrow(E\mathbf{x} \wedge A(\mathbf{x})))) \equiv \iota\mathbf{y}(M\mathbf{y} \wedge \forall\mathbf{x}(\mathbf{x}\in\mathbf{y}\leftrightarrow(E\mathbf{x} \wedge B(\mathbf{x}))))$. So by definition $\mathbf{x}{\cdot}A(\mathbf{x}) \equiv \mathbf{x}{\cdot}B(\mathbf{x})$.

PROOF OF (ii). Suppose $\forall\mathbf{x}(A(\mathbf{x})\leftrightarrow B(\mathbf{x}))$. Then $\iota\mathbf{y}(\mathbf{y}{=}\iota\mathbf{x}(E\mathbf{x} \wedge A(\mathbf{x})) \vee \mathbf{y}{=}\iota\mathbf{z}(M\mathbf{z} \wedge \forall\mathbf{x}(\mathbf{x}\in\mathbf{z}\leftrightarrow(E\mathbf{x} \wedge A(\mathbf{x}))))) \equiv \iota\mathbf{y}(\mathbf{y}{=}\iota\mathbf{x}(E\mathbf{x} \wedge B(\mathbf{x})) \vee \mathbf{y}{=}\iota\mathbf{z}(M\mathbf{z} \wedge \forall\mathbf{x}(\mathbf{x}\in\mathbf{z}\leftrightarrow(E\mathbf{x} \wedge B(\mathbf{x})))))$. So by definition $\mathbf{x}{:}A(\mathbf{x}) \equiv \mathbf{x}{:}B(\mathbf{x})$.

LEMMA 2. *Comprehension*

(i) $m\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z})) \leftrightarrow \mathsf{E}!(\mathbf{x}{\cdot}A(\mathbf{x}))$

(ii) $m\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z})) \leftrightarrow \mathbf{x}{:}A(\mathbf{x}) = \mathbf{x}{\cdot}A(\mathbf{x})$

(iii) $\exists_1\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z})) \leftrightarrow \mathbf{x}{:}A(\mathbf{x}) = \iota\mathbf{x}(E\mathbf{x} \wedge A(\mathbf{x}))$

(iv) $\exists\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z})) \leftrightarrow \mathsf{E}!(\mathbf{x}{:}A(\mathbf{x}))$

PROOF OF (i).

1 For the $\rightarrow$ half, suppose $m\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$. Then $E\mathbf{y} \wedge A(\mathbf{y})$ for some $\mathbf{y}$, whence $\mathbf{y}\in\mathbf{x}{\cdot}A(\mathbf{x})$ by axiom 1(v). Hence $\mathsf{E}!(\mathbf{x}{\cdot}A(\mathbf{x}))$ by axiom 1(i).

2 For the $\leftarrow$ half, suppose $\mathsf{E}!(\mathbf{x}{\cdot}A(\mathbf{x}))$. Then $\mathbf{x}{\cdot}A(\mathbf{x}) = \iota\mathbf{y}(M\mathbf{y} \wedge \forall\mathbf{x}(\mathbf{x}\in\mathbf{y}\leftrightarrow(E\mathbf{x} \wedge A(\mathbf{x}))))$ by the definition of $\mathbf{x}{\cdot}A(\mathbf{x})$, whence $M(\mathbf{x}{\cdot}A(\mathbf{x}))$. Hence $m\mathbf{z}(\mathbf{z}\in\mathbf{x}{\cdot}A(\mathbf{x}))$ by the definition of M, whence $m\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$.

PROOF OF (ii).

1 For the $\rightarrow$ half, suppose $m\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$. Then $\neg\exists_1\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$, whence $\neg\mathsf{E}!\iota\mathbf{x}(E\mathbf{x} \wedge A(\mathbf{x}))$. Hence by the strength of identity $(\mathbf{y} = \iota\mathbf{x}(E\mathbf{x} \wedge A(\mathbf{x})) \vee \mathbf{y}{=}\mathbf{x}{\cdot}A(\mathbf{x})) \leftrightarrow \mathbf{y}{=}\mathbf{x}{\cdot}A(\mathbf{x})$, whence $\iota\mathbf{y}(\mathbf{y} = \iota\mathbf{x}(E\mathbf{x} \wedge A(\mathbf{x})) \vee \mathbf{y}{=}\mathbf{x}{\cdot}A(\mathbf{x})) \equiv \iota\mathbf{y}(\mathbf{y}{=}\mathbf{x}{\cdot}A(\mathbf{x})) \equiv \mathbf{x}{\cdot}A(\mathbf{x})$. Hence $\mathbf{x}{:}A(\mathbf{x}){\equiv}\mathbf{x}{\cdot}A(\mathbf{x})$ by the definition of $\mathbf{x}{:}A(\mathbf{x})$. Since $m\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$ entails $\mathsf{E}!(\mathbf{x}{\cdot}A(\mathbf{x}))$ by lemma 2(i), it follows that $\mathbf{x}{:}A(\mathbf{x}){=}\mathbf{x}{\cdot}A(\mathbf{x})$.

2 For the $\leftarrow$ half, suppose $\mathbf{x}{:}A(\mathbf{x}){=}\mathbf{x}{\cdot}A(\mathbf{x})$. Then $\mathsf{E}!(\mathbf{x}{\cdot}A(\mathbf{x}))$ by the strength of identity, whence $m\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$ by lemma 2(i).

PROOF OF (iii).

1 For the $\rightarrow$ half, suppose $\exists_1\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$. Then $\neg\mathsf{E}!(\mathbf{x}{\cdot}A(\mathbf{x}))$ by lemma 2(i). Hence by the strength of identity $(\mathbf{y} = \iota\mathbf{x}(E\mathbf{x} \wedge A(\mathbf{x})) \vee \mathbf{y}{=}\mathbf{x}{\cdot}A(\mathbf{x})) \leftrightarrow \mathbf{y} = \iota\mathbf{x}(E\mathbf{x} \wedge A(\mathbf{x}))$, whence $\mathbf{x}{:}A(\mathbf{x}) \equiv \iota\mathbf{x}(E\mathbf{x} \wedge A(\mathbf{x}))$ by the definition of $\mathbf{x}{:}A(\mathbf{x})$. Since $\exists_1\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$, it follows that $\mathsf{E}!\iota\mathbf{x}(E\mathbf{x} \wedge A(\mathbf{x}))$, whence $\mathbf{x}{:}A(\mathbf{x}) = \iota\mathbf{x}(E\mathbf{x} \wedge A(\mathbf{x}))$.

2 For the $\leftarrow$ half, suppose $\mathbf{x}{:}A(\mathbf{x}) = \iota\mathbf{x}(E\mathbf{x} \wedge A(\mathbf{x}))$. Then $\mathsf{E}!\iota\mathbf{x}(E\mathbf{x} \wedge A(\mathbf{x}))$ by the strength of identity, whence $\exists_1\mathbf{x}(E\mathbf{x} \wedge A(\mathbf{x}))$.

PROOF OF (iv).

1 For the $\rightarrow$ half, suppose $\exists\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$. Then $m\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z})) \vee \exists_1\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$. If $m\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$, then $\mathsf{E}!(\mathbf{x}{:}A(\mathbf{x}))$ by lemma 2(ii) and the strength of identity. If $\exists_1\mathbf{z}(E\mathbf{z}\wedge A(\mathbf{z}))$, then $\mathsf{E}!(\mathbf{x}{:}A(\mathbf{x}))$ by lemma 2(iii) and the strength of identity.

2 For the $\leftarrow$ half, suppose $\mathsf{E}!(\mathbf{x}{:}A(\mathbf{x}))$. Then $\mathbf{x}{:}A(\mathbf{x}) = \imath\mathbf{x}(E\mathbf{x} \wedge A(\mathbf{x})) \vee \mathbf{x}{:}A(\mathbf{x}) = \mathbf{x}{\cdot}A(\mathbf{x})$ by the definition of $\mathbf{x}{:}A(\mathbf{x})$. If $\mathbf{x}{:}A(\mathbf{x}) = \imath\mathbf{x}(E\mathbf{x} \wedge A(\mathbf{x}))$, then $\exists_1\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$ by lemma 2(iii). If $\mathbf{x}{:}A(\mathbf{x}) = \mathbf{x}{\cdot}A(\mathbf{x})$, then $m\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$ by lemma 2(ii). So either way it follows that $\exists\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$.

LEMMA 3. *What there is*

(i) $\mathsf{E}!a \leftrightarrow Ua \vee Ma$

(ii) $Ma \leftrightarrow Sa \vee Ca$

(iii) $Ea \leftrightarrow m\mathbf{x}U\mathbf{x} \wedge (Ua \vee Sa)$

PROOF OF (i).

1 For the $\rightarrow$ half, suppose $\mathsf{E}!a$. Either Ma or $\neg Ma$. If Ma, a fortiori $Ua \vee Ma$. If $\neg Ma$ then by definition Ua, a fortiori $Ua \vee Ma$.

2 For the $\leftarrow$ half, suppose $Ua \vee Ma$. By the definition of U, if Ua then $\mathsf{E}!a$. By the definition of M, if Ma then $m\mathbf{x}\ \mathbf{x}{\in}a$, whence by axiom 1(i) $\mathsf{E}!a$.

PROOF OF (ii).

1 For the $\rightarrow$ half, suppose Ma. Either Sa or $\neg Sa$. If Sa, a fortiori $Sa \vee Ca$. If $\neg Sa$ then Ca by the definitions of S and C; a fortiori $Sa \vee Ca$.

2 For the $\leftarrow$ half, suppose $Sa \vee Ca$. If Sa then Ma by the definition of S. If Ca then Ma by the definition of C.

PROOF OF (iii).

1 For the $\rightarrow$ half, suppose Ea. Then for some $\mathbf{y}$, $a{\in}\mathbf{y}$, by the definition of E, whence $\mathsf{E}!a$ by axiom 1(i). Hence $\exists\mathbf{x}\ \mathbf{x}{\in}\mathbf{y}$, whence $M\mathbf{y}$ by axiom 1(iv). Hence $m\mathbf{x}U\mathbf{x}$ by axiom 2(ii). Also from Ea and $\mathsf{E}!a$ it follows that $Ua \vee Sa$ by lemmas 3(i) and 3(ii) and the definition of C.

2 For the $\leftarrow$ half, suppose $m\mathbf{x}U\mathbf{x} \wedge (Ua \vee Sa)$. If Ua, then Ea by axiom 2(i). If Sa, then Ea by the definition of S.

LEMMA 4. *Exhaustive description and reduction*

$a{=}\mathbf{x}{\cdot}A(\mathbf{x}) \leftrightarrow\ Ma \wedge \forall\mathbf{y}(\mathbf{y}{\in}a{\leftrightarrow}(E\mathbf{y} \wedge A(\mathbf{y})))$

PROOF.

1 For the $\rightarrow$ half, suppose $a{=}\mathbf{x}{\cdot}A(\mathbf{x})$. Then $a{=}\imath\mathbf{z}(M\mathbf{z} \wedge \forall\mathbf{y}(\mathbf{y}{\in}\mathbf{z}{\leftrightarrow}(E\mathbf{y} \wedge A(\mathbf{y}))))$ by the definition of $\mathbf{x}{\cdot}A(\mathbf{x})$, whence $Ma \wedge \forall\mathbf{y}(\mathbf{y}{\in}a{\leftrightarrow}(E\mathbf{y} \wedge A(\mathbf{y})))$.

2 For the $\leftarrow$ half, suppose $Ma \wedge \forall\mathbf{y}(\mathbf{y}{\in}a{\leftrightarrow}(E\mathbf{y} \wedge A(\mathbf{y})))$. Then by lemma 3(i) $\mathsf{E}!a$. For a reductio suppose $M\mathbf{x} \wedge \forall\mathbf{y}(\mathbf{y}{\in}\mathbf{x}{\leftrightarrow}(E\mathbf{y} \wedge A(\mathbf{y})))$ for some $\mathbf{x}{\neq}a$. Then $\forall\mathbf{y}(\mathbf{y}{\in}\mathbf{x}{\leftrightarrow}\mathbf{y}{\in}a)$, whence $\mathbf{x}{=}a$ by axiom 1(ii). Contradiction. Hence $a = \imath\mathbf{z}(M\mathbf{z} \wedge \forall\mathbf{y}(\mathbf{y}{\in}\mathbf{z}{\leftrightarrow}(E\mathbf{y} \wedge A(\mathbf{y}))))$, whence $a{=}\mathbf{x}{\cdot}A(\mathbf{x})$ by the definition of $\mathbf{x}{\cdot}A(\mathbf{x})$.

LEMMA 5. *Membership* (i) $a{\in}b \rightarrow Ea \wedge Mb$
(ii) $a{=}\mathbf{x}{\cdot}\mathbf{x}{\in}a{\leftrightarrow}Ma$
(iii) $a{=}\mathbf{x}{:}\mathbf{x}{\in}a{\leftrightarrow}Ma$

PROOF OF (i). Suppose $a{\in}b$. Then by axiom 1(i) $\mathsf{E}!b$. Hence $\exists\mathbf{x}\ a{\in}\mathbf{x}$, whence Ea by the definition of E. Also by axiom 1(i) $\mathsf{E}!a$. Hence $\exists\mathbf{x}\ \mathbf{x}{\in}b$, whence Mb by axiom 1(iv).

PROOF OF (ii).
1 For the $\rightarrow$ half, suppose $a{=}\mathbf{x}{\cdot}\mathbf{x}{\in}a$. Then Ma by lemma 4.
2 For the $\leftarrow$ half, suppose Ma. Since $\forall\mathbf{x}(\mathbf{x}{\in}a{\leftrightarrow}(E\mathbf{x} \wedge \mathbf{x}{\in}a))$ by lemma 5(i), it follows that $a{=}\mathbf{x}{\cdot}\mathbf{x}{\in}a$ by lemma 4.

PROOF OF (iii).
1 For the $\rightarrow$ half, suppose $a{=}\mathbf{x}{:}\mathbf{x}{\in}a$. Then $\mathsf{E}!(\mathbf{x}{:}\mathbf{x}{\in}a)$ by the strength of identity, whence $\exists\mathbf{z}\ \mathbf{z}{\in}a$ by lemma 2(iv). Hence Ma by axiom 1(iv).
2 For the $\leftarrow$ half, suppose Ma. Then $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}{\in}a)$ by the definition of M and lemma 5(i), whence $\mathbf{x}{:}\mathbf{x}{\in}a = \mathbf{x}{\cdot}\mathbf{x}{\in}a$ by lemma 2(ii). Hence $a{=}\mathbf{x}{:}\mathbf{x}{\in}a$ by lemma 5(ii).

LEMMA 6. *Submultitude* (i) $a{\subseteq}b{\rightarrow}Mb$
(ii) $Ma{\leftrightarrow}a{\subseteq}a$
(iii) $a{\subset}b{\rightarrow}\exists\mathbf{x}(\mathbf{x}{\notin}a \wedge \mathbf{x}{\in}b)$

PROOF OF (i). Suppose $a{\subseteq}b$. Then Ma by the definition of $\subseteq$, whence $m\mathbf{x}\ \mathbf{x}{\in}a$ by the definition of M. Also $\forall\mathbf{x}(\mathbf{x}{\in}a{\rightarrow}\mathbf{x}{\in}b)$ by the definition of $\subseteq$. Hence $m\mathbf{x}\ \mathbf{x}{\in}b$, whence Mb by the definition of M.

PROOF OF (ii).
1 For the $\rightarrow$ half suppose Ma. Since $\forall\mathbf{x}(\mathbf{x}{\in}a{\rightarrow}\mathbf{x}{\in}a)$, it follows by the definition of $\subseteq$ that $a{\subseteq}a$.
2 For the $\leftarrow$ half suppose $a{\subseteq}a$. Then Ma by the definition of $\subseteq$.

PROOF OF (iii). Suppose $a{\subset}b$. Then $a{\subseteq}b$ and $a{\neq}b$ by the definition of $\subset$, whence Ma and $\forall\mathbf{x}(\mathbf{x}{\in}a{\rightarrow}\mathbf{x}{\in}b)$ by the definition of $\subseteq$. Also by lemma 6(i), Mb. For a reductio suppose that $\forall\mathbf{x}(\mathbf{x}{\in}b{\rightarrow}\mathbf{x}{\in}a)$. Then by axiom 1(ii), $a{=}b$. Contradiction. Hence $\exists\mathbf{x}(\mathbf{x}{\notin}a \wedge \mathbf{x}{\in}b)$.

LEMMA 7. *Zilch* (i) $\neg\mathsf{E}!O$
(ii) $\neg UO \wedge \neg MO$
(iii) $\neg SO \wedge \neg CO$
(iv) $\neg EO$
(v) $a{\notin}O \wedge O{\notin}a$
(vi) $a{\not\subseteq}O \wedge O{\not\subseteq}a$
(vii) $\neg VO$
(viii) HO
(ix) $\neg\mathsf{E}!a \leftrightarrow a{\equiv}O$

PROOF OF (i). Since $\neg\exists\mathbf{x}\ \mathbf{x}\neq\mathbf{x}$, it follows that $\neg\mathrm{E}!\iota\mathbf{x}(\mathbf{x}\neq\mathbf{x})$, whence $\neg\mathrm{E}!O$ by the definition of O.

PROOF OF (ii). Immediate by lemmas 3(i) and 7(i).

PROOF OF (iii). Immediate by lemmas 3(ii) and 7(ii).

PROOF OF (iv). Immediate by lemmas 3(iii), 7(ii) and 7(iii).

PROOF OF (v). Immediate by axiom 1(i) and 7(i).

PROOF OF (vi). $a\not\subseteq O$ follows from lemmas 6(i) and 7(ii); $O\not\subseteq a$ follows from lemma 7(ii) and the definition of $\subseteq$.

PROOF OF (vii). Immediate by lemma 7(i), the definition of V, and the strength of identity.

PROOF OF (viii). Immediate by lemma 7(v) and the definition of H.

PROOF OF (ix).

1 For the $\rightarrow$ half suppose $\neg\mathrm{E}!a$. Then $a\equiv O$ by lemma 7(i) and the definition of $\equiv$.

2 For the $\leftarrow$ half suppose $a\equiv O$. By lemma 7(i) and the strength of identity, $a\neq O$, whence $\neg\mathrm{E}!a$ by the definition of $\equiv$.

LEMMA 8. *Levels* *Let Va, then* (i) Ma
(ii) Ea
(iii) Sa
(iv) $\forall\mathbf{y}(U\mathbf{y}\rightarrow\mathbf{y}\in a)$.

PROOF OF (i). From the hypothesis and the definitions of V, V_1 and V_2, it follows that either a=accum($\mathbf{z}$) or a=acc($\mathbf{z}$). If a=accum($\mathbf{z}$), then $a=\mathbf{x}\cdot(U\mathbf{x}\vee\mathbf{x}\in\mathbf{z}\vee\mathbf{x}\subseteq\mathbf{z})$ by the definition of accum, whence Ma by lemma 4. If a=acc($\mathbf{z}$), then $a=\mathbf{x}\cdot(U\mathbf{x}\vee\exists\mathbf{y}(\mathbf{y}\in\mathbf{z}\wedge(\mathbf{x}\in\mathbf{y}\vee\mathbf{x}\subseteq\mathbf{y})))$ by the definition of acc, whence Ma by lemma 4.

PROOF OF (ii). From the hypothesis it follows that $a\in\mathbf{x}$ for some $\mathbf{x}$ by axiom 2(v), whence Ea by the definition of E.

PROOF OF (iii). Immediate from the hypothesis by lemmas 8(i) and 8(ii), and the definition of S.

PROOF OF (iv). From the hypothesis and lemma 8(i) it follows that $\forall\mathbf{x}(U\mathbf{x}\rightarrow E\mathbf{x})$ by lemma 3(i) and axioms 2(i) and 2(ii). From the hypothesis and the definitions of V, V_1 and V_2, either a=accum($\mathbf{z}$) or a=acc($\mathbf{z}$). If a=accum($\mathbf{z}$), then $a=\mathbf{x}\cdot(U\mathbf{x}\vee\mathbf{x}\in\mathbf{z}\vee\mathbf{x}\subseteq\mathbf{z})$ by the definition of accum, whence $\forall\mathbf{y}(U\mathbf{y}\rightarrow\mathbf{y}\in a)$ by lemma 4. If a=acc($\mathbf{z}$), then $a=\mathbf{x}\cdot(U\mathbf{x}\vee\exists\mathbf{y}(\mathbf{y}\in\mathbf{z}\wedge(\mathbf{x}\in\mathbf{y}\vee\mathbf{x}\subseteq\mathbf{y})))$ by the definition of acc, whence $\forall\mathbf{y}(U\mathbf{y}\rightarrow\mathbf{y}\in a)$ by lemma 4.

LEMMA 9. *The first two levels*

(i) $V_1 \equiv \mathbf{z}{\cdot}U\mathbf{z} \equiv \mathrm{acc}(O) \equiv \mathrm{acc}(V_1)$

(ii) $\mathsf{E}!V_1 \leftrightarrow MV_1 \wedge \forall\mathbf{y}(\mathbf{y}{\in}V_1{\leftrightarrow}U\mathbf{y})$

(iii) $\exists\mathbf{x}M\mathbf{x} \leftrightarrow V_1 = \mathbf{z}{\cdot}U\mathbf{z}$

(iv) $V_2 \equiv \mathbf{z}{\cdot}(U\mathbf{z} \vee \mathbf{z}{\subseteq}V_1)$

(v) $\mathsf{E}!V_2 \leftrightarrow MV_2 \wedge \forall\mathbf{y}(\mathbf{y}{\in}V_2{\leftrightarrow}(U\mathbf{y} \vee \mathbf{y}{\subseteq}V_1))$

(vi) $\mathsf{E}!V_2 \leftrightarrow \mathsf{E}!V_1$

(vii) $\mathsf{E}!V_2 \leftrightarrow V_1{\in}V_2$

(viii) $V_1{\neq}V_2$

PROOF OF (i).

1 By definition $V_1 \equiv \mathrm{accum}(O) \equiv \mathbf{z}{\cdot}(U\mathbf{z} \vee \mathbf{z}{\in}O \vee \mathbf{z}{\subseteq}O)$. By lemmas 7(v) and 7(vi), $(U\mathbf{z} \vee \mathbf{z}{\in}O \vee \mathbf{z}{\subseteq}O){\leftrightarrow}U\mathbf{z}$, whence $V_1 \equiv \mathbf{z}{\cdot}U\mathbf{z}$ by lemma 1(i).

2 By the definition of acc, $\mathrm{acc}(O) \equiv \mathbf{z}{\cdot}(U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}{\in}O \wedge (\mathbf{z}{\in}\mathbf{y} \vee \mathbf{z}{\subseteq}\mathbf{y})))$. By lemma 7(v), $(U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}{\in}O \wedge (\mathbf{z}{\in}\mathbf{y} \vee \mathbf{z}{\subseteq}\mathbf{y}))){\leftrightarrow}U\mathbf{z}$, whence $\mathrm{acc}(O) \equiv \mathbf{z}{\cdot}U\mathbf{z}$ by lemma 1(i).

3 By the definition of acc, $\mathrm{acc}(V_1) \equiv \mathbf{z}{\cdot}(U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}{\in}V_1 \wedge (\mathbf{z}{\in}\mathbf{y} \vee \mathbf{z}{\subseteq}\mathbf{y})))$. If $\mathbf{y}{\in}V_1$ then $\mathsf{E}!V_1$ by axiom 1(i), whence $U\mathbf{y}$ by lemma 4. Hence $\neg M\mathbf{y}$ by the definition of U, whence by lemmas 5(i) and 6(i), $(U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}{\in}V_1 \wedge (\mathbf{z}{\in}\mathbf{y} \vee \mathbf{z}{\subseteq}\mathbf{y}))){\leftrightarrow}U\mathbf{z}$. Hence $\mathrm{acc}(V_1) \equiv \mathbf{z}{\cdot}U\mathbf{z}$ by lemma 1(i).

PROOF OF (ii).

1 For the $\rightarrow$ half suppose $\mathsf{E}!V_1$. Then $V_1 = \mathbf{z}{\cdot}U\mathbf{z}$ by lemma 9(i). Hence $MV_1 \wedge \forall\mathbf{y}(\mathbf{y}{\in}V_1{\leftrightarrow}(E\mathbf{y} \wedge U\mathbf{y}))$ by lemma 4, whence $MV_1 \wedge \forall\mathbf{y}(\mathbf{y}{\in}V_1{\leftrightarrow}U\mathbf{y}))$ by lemma 8(iv) and the definition of V.

2 For the $\leftarrow$ half suppose $MV_1 \wedge \forall\mathbf{y}(\mathbf{y}{\in}V_1{\leftrightarrow}U\mathbf{y})$. Then $\mathsf{E}!V_1$ by lemma 3(i).

PROOF OF (iii).

1 For the $\rightarrow$ half suppose $\exists\mathbf{x}M\mathbf{x}$. Then $m\mathbf{x}(E\mathbf{x} \wedge U\mathbf{x})$ by axioms 2(i) and 2(ii). Hence $\mathsf{E}!(\mathbf{z}{\cdot}U\mathbf{z})$ by lemma 2(i), whence $V_1 = \mathbf{z}{\cdot}U\mathbf{z}$ by lemma 9(i).

2 For the $\leftarrow$ half, suppose $V_1 = \mathbf{z}{\cdot}U\mathbf{z}$. Then $\mathsf{E}!(\mathbf{z}{\cdot}U\mathbf{z})$ by the strength of identity, whence $\exists\mathbf{x}M\mathbf{x}$ by the definition of $\mathbf{z}{\cdot}U\mathbf{z}$.

PROOF OF (iv). By definition $V_2 \equiv \mathrm{accum}(V_1) \equiv \mathbf{z}{\cdot}(U\mathbf{z} \vee \mathbf{z}{\in}V_1 \vee \mathbf{z}{\subseteq}V_1)$. Suppose $\neg\mathsf{E}!V_1$ Then $\neg\exists\mathbf{y}\ \mathbf{y}{\in}V_1$ by axiom 1(i). Suppose instead $\mathsf{E}!V_1$. Then $\mathbf{z}{\in}V_1{\leftrightarrow}U\mathbf{z}$ by lemma 9(ii). So either way $(U\mathbf{z} \vee \mathbf{z}{\in}V_1 \vee \mathbf{z}{\subseteq}V_1){\leftrightarrow}(U\mathbf{z} \vee \mathbf{z}{\subseteq}V_1)$. Hence $V_2 \equiv \mathbf{z}{\cdot}(U\mathbf{z} \vee \mathbf{z}{\subseteq}V_1)$ by lemma 1(i).

PROOF OF (v).

1 For the $\rightarrow$ half suppose $\mathsf{E}!V_2$. Then $V_2{=}\mathbf{z}{\cdot}(U\mathbf{z} \vee \mathbf{z}{\subseteq}V_1)$ by lemma 9(iv). Hence $MV_2 \wedge \forall\mathbf{y}(\mathbf{y}{\in}V_2{\leftrightarrow}(E\mathbf{y} \wedge (U\mathbf{y} \vee \mathbf{y}{\subseteq}V_1)))$ by lemma 4, whence $\forall\mathbf{y}(\mathbf{y}{\in}V_2{\leftrightarrow}(U\mathbf{y} \vee \mathbf{y}{\subseteq}V_1))$ by lemmas 8(iii) and 8(iv), axiom 2(iii), and the definitions of S and V.

2 For the $\leftarrow$ half suppose $MV_2 \wedge \forall\mathbf{y}(\mathbf{y}{\in}V_2{\leftrightarrow}(U\mathbf{y} \vee \mathbf{y}{\subseteq}V_1))$. Then $\mathsf{E}!V_2$ by lemma 3(i).

PROOF OF (vi).

1 For the $\rightarrow$ half suppose $\mathsf{E}!V_2$. Then MV_2 by lemma 9(v), whence $V_1 = \mathbf{z}{\cdot}U\mathbf{z}$ by lemma 9(iii), whence $\mathsf{E}!V_1$ by the strength of identity.

2 For the $\leftarrow$ half suppose $\mathsf{E}!V_1$. Then $V_1 = \mathbf{z}{\cdot}U\mathbf{z}$ by lemma 9(i), whence $m\mathbf{z}(E\mathbf{z} \wedge U\mathbf{z})$ by lemma 2(i), a fortiori $m\mathbf{z}(E\mathbf{z} \wedge (U\mathbf{z} \vee \mathbf{z}{\subseteq}V_1)$. Hence $\mathsf{E}!\mathbf{z}{\cdot}(U\mathbf{z} \vee \mathbf{z}{\subseteq}V_1)$ by lemma 2(i), whence $\mathsf{E}!V_2$ by lemma 9(iv).

PROOF OF (vii).

1 For the $\rightarrow$ half suppose $\mathsf{E}!V_2$. Then $\mathsf{E}!V_1$ by lemma 9(vi), whence MV_1 by lemma 9(ii). Hence $V_1{\subseteq}V_1$ by lemma 6(ii), whence $V_1{\in}V_2$ by lemma 9(v).

2 For the $\leftarrow$ half suppose $V_1{\in}V_2$. Then $\mathsf{E}!V_2$ by axiom 1(i).

PROOF OF (viii). For a reductio suppose $V_1{=}V_2$. Then $\mathsf{E}!V_1$ and $\mathsf{E}!V_2$ by the strength of identity. Hence $V_1{\in}V_2$ by lemma 9(vii), whence $V_1{\in}V_1$. By lemma 9(ii) UV_1, whence $\neg MV_1$ by the definition of U. But MV_1 also by lemma 9(ii). Contradiction. Hence $V_1{\neq}V_2$.

THEOREM 1. *Separation*

(i) $S\mathbf{x} \wedge m\mathbf{y}(\mathbf{y}{\in}\mathbf{x} \wedge A(\mathbf{y})) \rightarrow S(\mathbf{z}{\cdot}\mathbf{z}{\in}\mathbf{x} \wedge A(\mathbf{z}))$

(ii) $S\mathbf{x} \leftrightarrow \exists\mathbf{u}\ \mathbf{x}{\subseteq}\mathbf{u}$

(iii) $S(\mathbf{x}{\cdot}A(\mathbf{x})) \leftrightarrow (m\mathbf{y}(E\mathbf{y} \wedge A(\mathbf{y})) \wedge \exists\mathbf{u}\forall\mathbf{z}((E\mathbf{z} \wedge A(\mathbf{z})){\rightarrow}\mathbf{z}{\in}\mathbf{u}))$

PROOF OF (i). Suppose $S\mathbf{x} \wedge m\mathbf{y}(\mathbf{y}{\in}\mathbf{x} \wedge A(\mathbf{y}))$. Then $m\mathbf{y}(E\mathbf{y} \wedge \mathbf{y}{\in}\mathbf{x} \wedge A(\mathbf{y}))$ by lemma 5(i), whence $\mathsf{E}!(\mathbf{z}{\cdot}\mathbf{z}{\in}\mathbf{x} \wedge A(\mathbf{z}))$ by lemma 2(i). Hence $M(\mathbf{z}{\cdot}\mathbf{z}{\in}\mathbf{x} \wedge A(\mathbf{z})) \wedge \forall\mathbf{y}(\mathbf{y}{\in}\mathbf{z}{\cdot}(\mathbf{z}{\in}\mathbf{x} \wedge A(\mathbf{z})){\rightarrow}\mathbf{y}{\in}\mathbf{x})$ by the definition of $\mathbf{z}{\cdot}(\mathbf{z}{\in}\mathbf{x} \wedge A(\mathbf{z}))$, whence $\mathbf{z}{\cdot}(\mathbf{z}{\in}\mathbf{x} \wedge A(\mathbf{z})) \subseteq \mathbf{x}$ by the definition of $\subseteq$. Hence $S(\mathbf{z}{\cdot}\mathbf{z}{\in}\mathbf{x} \wedge A(\mathbf{z}))$ by axiom 2(iii).

PROOF OF (ii). The $\rightarrow$ half is axiom 2(iv). For the $\leftarrow$ half suppose $\mathbf{x}{\subseteq}\mathbf{u}$ for some level $\mathbf{u}$. Then $S\mathbf{u}$ by lemma 8(iii), whence $S\mathbf{x}$ by axiom 2(iii).

PROOF OF (iii).

1 For the $\rightarrow$ half suppose $S(\mathbf{x}{\cdot}A(\mathbf{x}))$. Then $\mathsf{E}!(\mathbf{x}{\cdot}A(\mathbf{x}))$ by lemmas 3(i) and 3(ii), whence $M(\mathbf{x}{\cdot}A(\mathbf{x})) \wedge \forall\mathbf{y}(\mathbf{y}{\in}\mathbf{x}{\cdot}A(\mathbf{x}) \leftrightarrow E\mathbf{y} \wedge A(\mathbf{y}))$ by lemma 4. Hence $m\mathbf{y}\ \mathbf{y}{\in}\mathbf{x}{\cdot}A(\mathbf{x})$ by the definition of M, whence $m\mathbf{y}(E\mathbf{y} \wedge A(\mathbf{y}))$. By axiom 2(iv), $\exists\mathbf{u}\ \mathbf{x}{\cdot}A(\mathbf{x}){\subseteq}\mathbf{u}$. Hence $\exists\mathbf{u}\forall\mathbf{z}(\mathbf{z}{\in}\mathbf{x}{\cdot}A(\mathbf{x}) \rightarrow \mathbf{z}{\in}\mathbf{u})$ by the definition of $\subseteq$, whence $\exists\mathbf{u}\forall\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}) \rightarrow \mathbf{z}{\in}\mathbf{u})$.

2 For the $\leftarrow$ half suppose $m\mathbf{y}(E\mathbf{y} \wedge A(\mathbf{y}))$ and $\forall\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}) \rightarrow \mathbf{z}{\in}\mathbf{u})$ for some level $\mathbf{u}$. Then $\mathsf{E}!(\mathbf{x}{\cdot}A(\mathbf{x}))$ by lemma 2(i). Hence $M(\mathbf{x}{\cdot}A(\mathbf{x})) \wedge \forall\mathbf{z}(\mathbf{z}{\in}\mathbf{x}{\cdot}A(\mathbf{x}) \leftrightarrow E\mathbf{z} \wedge A(\mathbf{z}))$ by lemma 4, whence $M(\mathbf{x}{\cdot}A(\mathbf{x})) \wedge \forall\mathbf{z}(\mathbf{z}{\in}\mathbf{x}{\cdot}A(\mathbf{x}) \rightarrow \mathbf{z}{\in}\mathbf{u})$. Hence $\exists\mathbf{u}\ \mathbf{x}{\cdot}A(\mathbf{x}){\subseteq}\mathbf{u}$ by the definition of $\subseteq$, whence $S(\mathbf{x}{\cdot}A(\mathbf{x}))$ by theorem 1(ii).

THEOREM 2. *Intersection*

(i) $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})\leftrightarrow\mathsf{E!}(\mathbf{x}\cap\mathbf{y})$

(ii) *Let* $\exists_1\mathbf{z}(\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})$, *then* $E(\mathbf{x}\cap\mathbf{y})$.

(iii) *Let* $m\mathbf{z}(\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})$, *then* $M(\mathbf{x}\cap\mathbf{y})$.

(iv) *Let* $m\mathbf{z}(\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})$ *and* $(S\mathbf{x} \vee S\mathbf{y})$, *then* $S(\mathbf{x}\cap\mathbf{y})$.

PROOF OF (i). By lemma 5(i), $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})\leftrightarrow\exists\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})$, whence $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})\leftrightarrow\mathsf{E!}\mathbf{z}{:}(\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})$ by lemma 2(iv). Hence $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})\leftrightarrow\mathsf{E!}(\mathbf{x}\cap\mathbf{y})$ by the definition of $\cap$.

PROOF OF (ii). From the hypothesis it follows that $\exists_1\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})$ by lemma 5(i), whence $\mathbf{z}{:}(\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})=\iota\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})$ by lemma 2(iii). Hence $E(\mathbf{z}{:}\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})$, whence $E(\mathbf{x}\cap\mathbf{y})$ by the definition of $\cap$.

PROOF OF (iii). From the hypothesis it follows that $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})$ by lemma 5(i), whence $\mathbf{z}{:}(\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y}) = \mathbf{z}\cdot(\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})$ by lemma 2(ii). Hence $M(\mathbf{z}{:}\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})$ by lemma 4, whence $M(\mathbf{x}\cap\mathbf{y})$ by the definition of $\cap$.

PROOF OF (iv). From the hypothesis it follows that $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})$ by lemma 5(i); a fortiori $m\mathbf{z}(\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})$. If $S\mathbf{x}$ then $S(\mathbf{z}\cdot\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})$ by theorem 1(i). If $S\mathbf{y}$ then $S(\mathbf{z}\cdot\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})$ by theorem 1(i). So either way $S(\mathbf{z}\cdot\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})$. Hence $S(\mathbf{x}\cap\mathbf{y})$ by lemma 2(ii) and the definition of $\cap$.

THEOREM 3. *Histories I* *Let* $\mathbf{v}=\mathrm{acc}(\mathbf{h})$, *then* (i) $\mathbf{x}\in\mathbf{h} \rightarrow \mathbf{x}\in\mathbf{v}$
(ii) $\mathbf{v}\neq V_1 \rightarrow M\mathbf{h} \wedge \mathbf{h}\neq V_1$.

PROOF OF (i). Suppose $\mathbf{x}\in\mathbf{h}$. Then $E\mathbf{x}$ by lemma 5(i). Also by axiom 1(i) it follows that $\mathsf{E!}\mathbf{x}$, whence $E\mathbf{x} \wedge (U\mathbf{x} \vee M\mathbf{x})$ by lemma 3(i). Suppose $E\mathbf{x} \wedge U\mathbf{x}$. Then a fortiori $E\mathbf{x} \wedge (U\mathbf{x} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{h} \wedge (\mathbf{x}\in\mathbf{y} \vee \mathbf{x}\subseteq\mathbf{y})))$. Suppose instead $E\mathbf{x} \wedge M\mathbf{x}$. Then by lemma 6(ii), $\mathbf{x}\subseteq\mathbf{x}$. Since $\mathbf{x}\in\mathbf{h}$, it follows that $E\mathbf{x} \wedge (U\mathbf{x} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{h} \wedge (\mathbf{x}\in\mathbf{y} \vee \mathbf{x}\subseteq\mathbf{y})))$. So either way $E\mathbf{x} \wedge (U\mathbf{x} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{h} \wedge (\mathbf{x}\in\mathbf{y} \vee \mathbf{x}\subseteq\mathbf{y})))$. From the hypothesis by the definition of acc, $\mathbf{v}=\mathbf{x}\cdot(U\mathbf{x} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{h} \wedge (\mathbf{x}\in\mathbf{y} \vee \mathbf{x}\subseteq\mathbf{y})))$, whence $\mathbf{x}\in\mathbf{v}$ by lemma 4.

PROOF OF (ii). Suppose $\mathbf{v}\neq V_1$. For a reductio suppose $\mathbf{h}\equiv O$. Then from the hypothesis by lemma 9(i) $\mathbf{v} = \mathrm{acc}(O) = V_1$. Contradiction. Hence $S\mathbf{h}$ by the definition of H, whence $M\mathbf{h}$ by the definition of S. For a reductio suppose $\mathbf{h}=V_1$. Then from the hypothesis by lemma 9(i) $\mathbf{v} = \mathrm{acc}(V_1) = V_1$. Contradiction. Hence $\mathbf{h}\neq V_1$.

THEOREM 4. *Histories II* *Let* $\mathbf{h}\neq V_1$ *and* $\mathbf{x}\in\mathbf{h}$, *then* $M\mathbf{x}$.

PROOF. Since by hypothesis $\mathbf{h}\neq V_1$ and $\mathbf{x}\in\mathbf{h}$, it follows that either $\mathbf{x}=\mathrm{accum}(\mathbf{z})$ or $\mathbf{x}=\mathrm{acc}(\mathbf{z})$. by the definitions of H, whence $M\mathbf{x}$ by lemma 4 and the definitions of accum and acc.

THEOREM 5. *Histories III* *Let* $\mathbf{h}\neq V_1$, $\mathbf{x}\in\mathbf{h}$, $\mathbf{x}\neq V_1$ *and* $\mathbf{x}\neq V_2$. *Then*
(i) $\mathbf{x}=\mathrm{acc}(\mathbf{h}\cap\mathbf{x})$
(ii) $\mathsf{E}!(\mathbf{h}\cap\mathbf{x})$
(iii) $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\in\mathbf{h} \wedge \mathbf{z}\in\mathbf{x})$
(iv) $S(\mathbf{h}\cap\mathbf{x})$.

PROOF OF (i). Since by hypothesis $\mathbf{h}\neq V_1$ and $\mathbf{x}\in\mathbf{h}$ and $\mathbf{x}\neq V_1$ and $\mathbf{x}\neq V_2$, it follows by the definition of H that $\mathbf{x}=\mathrm{acc}(\mathbf{h}\cap\mathbf{x})$.

PROOF OF (ii). For a reductio suppose that $\mathbf{h}\cap\mathbf{x}\equiv O$. Then by theorem 5(i) $\mathbf{x}=\mathrm{acc}(O)$, whence $\mathbf{x}=V_1$ by lemma 9(i). Contradiction. Hence $\mathsf{E}!(\mathbf{h}\cap\mathbf{x})$.

PROOF OF (iii).

1 By theorem 5(ii) $\mathsf{E}!(\mathbf{h}\cap\mathbf{x})$ and so by the definition of $\cap$ and lemma 2(iv), either $\exists_1\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\in\mathbf{h} \wedge \mathbf{z}\in\mathbf{x})$ or $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\in\mathbf{h} \wedge \mathbf{z}\in\mathbf{x})$. For a reductio suppose that $E\mathbf{z_1} \wedge \mathbf{z_1}\in\mathbf{h} \wedge \mathbf{z_1}\in\mathbf{x}$ for some unique $\mathbf{z_1}$. Then by the definition of $\cap$ and lemma 2(iii), $\mathbf{h}\cap\mathbf{x}=\mathbf{z_1}$. Hence by theorem 5(i) $\mathbf{x}=\mathrm{acc}(\mathbf{z_1})$.

2 For a reductio suppose $\mathbf{h}\cap\mathbf{z_1}\equiv O$. Since $\mathbf{h}\neq V_1$ and $\mathbf{z_1}\in\mathbf{h}$, it follows by the definition of H that $\mathbf{z_1}=\mathrm{accum}(\mathbf{h}\cap\mathbf{z_1})$ or $\mathbf{z_1}=\mathrm{acc}(\mathbf{h}\cap\mathbf{z_1})$. Suppose $\mathbf{z_1}=\mathrm{accum}(\mathbf{h}\cap\mathbf{z_1})$. Then $\mathbf{z_1} = \mathrm{accum}(O) = V_1$ by the definition of V_1. Suppose $\mathbf{z_1}=\mathrm{acc}(\mathbf{h}\cap\mathbf{z_1})$. Then $\mathbf{z_1} = \mathrm{acc}(O) = V_1$ by lemma 9(i). So either way $\mathbf{z_1}=V_1$. Hence $\mathbf{x} = \mathrm{acc}(\mathbf{z_1}) = \mathrm{acc}(V_1)$, whence by lemma 9(i), $\mathbf{x}=V_1$. Contradiction. Hence $\mathsf{E}!(\mathbf{h}\cap\mathbf{z_1})$.

3 Returning to the reductio initiated in step 1, since $\mathsf{E}!(\mathbf{h}\cap\mathbf{z_1})$ it follows by the definition of $\cap$ and lemma 2(iv) that $E\mathbf{z_2} \wedge \mathbf{z_2}\in\mathbf{h} \wedge \mathbf{z_2}\in\mathbf{z_1}$ for some $\mathbf{z_2}$, whence $\mathbf{z_2}\neq\mathbf{z_1}$ by axiom 1(iii). Now by the definition of acc, $\mathbf{x} = \mathrm{acc}(\mathbf{z_1}) = \mathbf{z}\cdot(U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{z_1} \wedge (\mathbf{z}\in\mathbf{y} \vee \mathbf{z}\subseteq\mathbf{y})))$. Since $\mathbf{h}\neq V_1$ and $\mathbf{z_2}\in\mathbf{h}$, it follows by theorem 4 that $M\mathbf{z_2}$. Hence by lemma 6(ii), $\mathbf{z_2}\subseteq\mathbf{z_2}$. Since $\mathbf{z_2}\in\mathbf{z_1}$ and $\mathbf{z_2}\subseteq\mathbf{z_2}$ and $E\mathbf{z_2}$, it follows that $\mathbf{z_2}\in\mathbf{x}$ by lemma 4. Hence $\mathbf{z_1}\in\mathbf{h}$ and $\mathbf{z_1}\in\mathbf{x}$ and $\mathbf{z_2}\in\mathbf{h}$ and $\mathbf{z_2}\in\mathbf{x}$ and $\mathbf{z_2}\neq\mathbf{z_1}$. Contradiction. Hence $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\in\mathbf{h} \wedge \mathbf{z}\in\mathbf{x})$.

PROOF OF (iv). Since by hypothesis $\mathbf{x}\in\mathbf{h}$, it follows by axiom 1(i) that $\mathsf{E}!\mathbf{h}$, whence $S\mathbf{h}$ by the definition of H. Since $m\mathbf{z}(\mathbf{z}\in\mathbf{h} \wedge \mathbf{z}\in\mathbf{x})$ by theorem 5(iii), it follows that $S(\mathbf{h}\cap\mathbf{x})$ by theorem 2(iv).

THEOREM 6. *Histories IV* *Let* $\mathbf{h}\neq V_1$ *and* $\mathbf{x}\in\mathbf{h}$, *then* $H(\mathbf{h}\cap\mathbf{x})$ *and* $V\mathbf{x}$.

PROOF. By hypothesis $\mathbf{x}\in\mathbf{h}$, whence $\mathsf{E}!\mathbf{x} \wedge \mathsf{E}!\mathbf{h}$ by axiom 1(i), and $M\mathbf{h}$ by lemma 5(i). Hence $\exists\mathbf{x}M\mathbf{x}$, whence $V_1 = \mathbf{z}\cdot U\mathbf{z}$ by lemma 9(iii). Hence $\mathsf{E}!V_1$ by the strength of identity, whence $\mathsf{E}!V_2$ by lemma 9(vi). It also follows from $\exists\mathbf{x}M\mathbf{x}$ that $\forall\mathbf{x}(U\mathbf{x}\rightarrow E\mathbf{x})$ by axioms 2(i) and 2(ii). We consider three cases separately: (i) $\mathbf{x}=V_1$, (ii) $\mathbf{x}=V_2$, (iii) $\mathbf{x}\neq V_1$ and $\mathbf{x}\neq V_2$.

CASE (i) $\mathbf{x}=V_1$

Since $\mathbf{x}=V_1$ it follows that $V\mathbf{x}$ by the definition of V, and also that $\mathbf{x} = \mathbf{z}{\cdot}U\mathbf{z}$. Hence $\mathbf{y}\in\mathbf{x}\rightarrow U\mathbf{y}$ by lemma 4. Also $\mathbf{y}\in\mathbf{h}\rightarrow\neg U\mathbf{y}$ by theorem 4 and the definition of U, whence $\neg\exists\mathbf{z}(\mathbf{z}\in\mathbf{h} \wedge \mathbf{z}\in\mathbf{x})$. Hence $\mathbf{h}\cap\mathbf{x}\equiv O$ by theorem 2(i) and lemma 7(ix), whence $H(\mathbf{h}\cap\mathbf{x})$ by lemma 7(viii).

CASE (ii) $\mathbf{x}=V_2$

1 Since $\mathbf{x}=V_2$ it follows that $V\mathbf{x}$ by the definition of V. Since $\mathbf{h}\neq V_1$ and $\mathbf{x}\in\mathbf{h}$, then $\mathbf{x}=\text{accum}(\mathbf{h}\cap\mathbf{x})$ by the definition of H.

2 For a reductio suppose that $\mathbf{h}\cap\mathbf{x}\equiv O$. Then $\mathbf{x} = \text{accum}(O) = V_1$ by the definition of V_1. But by lemma 9(viii), $V_1\neq V_2$. Contradiction. Hence $\mathsf{E}!\mathbf{h}\cap\mathbf{x}$.

3 Since $\mathbf{h}\cap\mathbf{x} =_{df} \mathbf{y}{:}(\mathbf{y}\in\mathbf{h} \wedge \mathbf{y}\in\mathbf{x})$, it follows that $\mathsf{E}!(\mathbf{y}{:}\mathbf{y}\in\mathbf{h} \wedge \mathbf{y}\in\mathbf{x})$, whence by lemma 2(iv), $E\mathbf{y} \wedge \mathbf{y}\in\mathbf{h} \wedge \mathbf{y}\in\mathbf{x}$ for some $\mathbf{y}$. We shall prove that there is exactly one such $\mathbf{y}$, namely V_1.

4 Since $\mathbf{x}=V_2$, it follows by lemma 9(iv) that $\mathbf{x} = \mathbf{z}{\cdot}(U\mathbf{z} \vee \mathbf{z}\subseteq V_1) = \mathbf{z}{\cdot}(U\mathbf{z} \vee \mathbf{z}\subseteq\mathbf{z_1}{\cdot}U\mathbf{z_1})$. Since $\mathbf{y}\in\mathbf{x}$, it follows by lemma 4 that $U\mathbf{y} \vee \mathbf{y}\subseteq\mathbf{z_1}{\cdot}U\mathbf{z_1}$. Since $\mathbf{y}\in\mathbf{h}$, it follows by theorem 4 and the definition of U that $\neg U\mathbf{y}$. Hence $\mathbf{y}\subseteq\mathbf{z_1}{\cdot}U\mathbf{z_1}$.

5 For a reductio suppose that $\mathbf{y}\subset\mathbf{z_1}{\cdot}U\mathbf{z_1}$. Then by lemma 6(iii), $\mathbf{y_1}\in\mathbf{z_1}{\cdot}U\mathbf{z_1}$ and $\mathbf{y_1}\notin\mathbf{y}$ for some $\mathbf{y_1}$, whence by lemma 4, $U\mathbf{y_1}$ and $\mathbf{y_1}\notin\mathbf{y}$. By lemmas 9(ii) and 9(v), $\forall\mathbf{x}(U\mathbf{x}\rightarrow\mathbf{x}\in V_1)$ and $\forall\mathbf{x}(U\mathbf{x}\rightarrow\mathbf{x}\in V_2)$, whence $\mathbf{y}\neq V_1$ and $\mathbf{y}\neq V_2$. Since $\mathbf{h}\neq V_1$ and $\mathbf{y}\in\mathbf{h}$ and $\mathbf{y}\neq V_1$ and $\mathbf{y}\neq V_2$, it follows that $\mathbf{y} = \text{acc}(\mathbf{h}\cap\mathbf{y}) = \mathbf{x}{\cdot}(U\mathbf{x} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{h}\cap\mathbf{y} \wedge (\mathbf{x}\in\mathbf{y} \vee \mathbf{x}\subseteq\mathbf{y})))$ by the definitions of H and acc. Since $U\mathbf{y_1}$ and $\forall\mathbf{x}(U\mathbf{x}\rightarrow E\mathbf{x})$, it follows by lemma 4 that $\mathbf{y_1}\in\mathbf{y}$. Contradiction. Hence $\mathbf{y}\not\subset\mathbf{z_1}{\cdot}U\mathbf{z_1}$. Since $\mathbf{y}\subseteq \mathbf{z_1}{\cdot}U\mathbf{z_1}$, it follows by the definition of $\subset$ that $\mathbf{y} = \mathbf{z_1}{\cdot}U\mathbf{z_1} = V_1$, whence $V_1 = \iota\mathbf{y}(E\mathbf{y} \wedge \mathbf{y}\in\mathbf{h} \wedge \mathbf{y}\in\mathbf{x})$. Hence $\exists_1\mathbf{y}(E\mathbf{y} \wedge \mathbf{y}\in\mathbf{h} \wedge \mathbf{y}\in\mathbf{x})$, whence by lemma 2(iii) and the definition of $\cap$, $\mathbf{h}\cap\mathbf{x} = V_1$. Also SV_1 by the definition of V and lemma 8(iii). Hence $H(\mathbf{h}\cap\mathbf{x})$ by the definition of H.

CASE (iii) $\mathbf{x}\neq V_1$ and $\mathbf{x}\neq V_2$

1 By theorems 5(i), 5(ii), 5(iii) and 5(iv), $\mathbf{x}=\text{acc}(\mathbf{h}\cap\mathbf{x})$, $\mathsf{E}!(\mathbf{h}\cap\mathbf{x})$, $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\in\mathbf{h} \wedge \mathbf{z}\in\mathbf{x})$, and $S(\mathbf{h}\cap\mathbf{x})$. By the definition of $\cap$ and lemma 2(ii), $\mathbf{h}\cap\mathbf{x} = \mathbf{z}{\cdot}(\mathbf{z}\in\mathbf{h} \wedge \mathbf{z}\in\mathbf{x})$. Hence by lemmas 4 and 5(i) $\mathbf{y}\in\mathbf{h}\cap\mathbf{x} \leftrightarrow \mathbf{y}\in\mathbf{h} \wedge \mathbf{y}\in\mathbf{x}$, whence $m\mathbf{z}\ \mathbf{z}\in\mathbf{h}\cap\mathbf{x}$.

2 Consider an arbitrary $\mathbf{z_1}$ such that $\mathbf{z_1}\in\mathbf{h}\cap\mathbf{x}$. Then $\mathbf{z}\in\mathbf{z_1}\rightarrow\exists\mathbf{y}(\mathbf{y}\in\mathbf{h}\cap\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})$, a fortiori $\mathbf{z}\in\mathbf{z_1}\rightarrow(U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{h}\cap\mathbf{x} \wedge (\mathbf{z}\in\mathbf{y} \vee \mathbf{z}\subseteq\mathbf{y})))$, whence by lemma 5(i), $\mathbf{z}\in\mathbf{z_1}\rightarrow(E\mathbf{z} \wedge (U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{h}\cap\mathbf{x} \wedge (\mathbf{z}\in\mathbf{y} \vee \mathbf{z}\subseteq\mathbf{y}))))$. By the definition of acc, $\mathbf{x} = \text{acc}(\mathbf{h}\cap\mathbf{x}) = \mathbf{z}{\cdot}(U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{h}\cap\mathbf{x} \wedge (\mathbf{z}\in\mathbf{y} \vee \mathbf{z}\subseteq\mathbf{y})))$. Hence by lemma 4, $\mathbf{z}\in\mathbf{z_1}\rightarrow\mathbf{z}\in\mathbf{x}$. By the definition of $\cap$, $(\mathbf{h}\cap\mathbf{x})\cap\mathbf{z_1} \equiv \mathbf{y}{:}(\mathbf{y}\in\mathbf{h}\cap\mathbf{x} \wedge \mathbf{y}\in\mathbf{z_1})$, whence $(\mathbf{h}\cap\mathbf{x})\cap\mathbf{z_1} \equiv \mathbf{y}{:}(\mathbf{y}\in\mathbf{h} \wedge \mathbf{y}\in\mathbf{x} \wedge \mathbf{y}\in\mathbf{z_1})$ by lemma 1(ii). Since $\mathbf{z}\in\mathbf{z_1}\rightarrow\mathbf{z}\in\mathbf{x}$, it follows by lemma 1(ii) that $(\mathbf{h}\cap\mathbf{x})\cap\mathbf{z_1} \equiv \mathbf{y}{:}(\mathbf{y}\in\mathbf{h} \wedge \mathbf{y}\in\mathbf{z_1})$, whence $(\mathbf{h}\cap\mathbf{x})\cap\mathbf{z_1} \equiv \mathbf{h}\cap\mathbf{z_1}$ by the definition of $\cap$.

3 Since $\mathbf{z_1}\in\mathbf{h}\cap\mathbf{x}$, it follows that $\mathbf{z_1}\in\mathbf{h}$. Suppose $\mathbf{z_1}=V_1 \vee \mathbf{z_1}=V_2$. Since $\mathbf{h}\neq V_1$ and $\mathbf{z_1}\in\mathbf{h}$, then $\mathbf{z_1} = \text{accum}(\mathbf{h}\cap\mathbf{z_1})$ by the definition of H, whence $\mathbf{z_1} = \text{accum}((\mathbf{h}\cap\mathbf{x})\cap\mathbf{z_1})$. Suppose instead that $\mathbf{z_1}\neq V_1 \wedge \mathbf{z_1}\neq V_2$. Then by the definition of H it follows that $\mathbf{z_1} = \text{acc}(\mathbf{h}\cap\mathbf{z_1})$, whence $\mathbf{z_1} = \text{acc}((\mathbf{h}\cap\mathbf{x})\cap\mathbf{z_1})$. Since $\mathbf{z_1}$ is arbitrary, we can generalize to get $\forall\mathbf{y}(\mathbf{y}\in\mathbf{h}\cap\mathbf{x}\rightarrow (\mathbf{y}=V_1 \vee \mathbf{y}=V_2 \rightarrow \mathbf{y}=\text{accum}((\mathbf{h}\cap\mathbf{x})\cap\mathbf{y})) \wedge (\mathbf{y}\neq V_1 \wedge \mathbf{y}\neq V_2 \rightarrow \mathbf{y}=\text{acc}((\mathbf{h}\cap\mathbf{x})\cap\mathbf{y})))$, which together with $S(\mathbf{h}\cap\mathbf{x})$ entails $H(\mathbf{h}\cap\mathbf{x})$ by the definition of H.

4 Since $\mathsf{E}!(\mathbf{h}\cap\mathbf{x})$, $H(\mathbf{h}\cap\mathbf{x})$, and $\mathbf{x}=\text{acc}(\mathbf{h}\cap\mathbf{x})$, it follows that $V\mathbf{x}$ by the definition of V.

THEOREM 7. $\in$ *is well-founded on any history* *Let* $\mathbf{x}\subseteq\mathbf{h}$, *then* $\exists\mathbf{y}(\mathbf{y}\in\mathbf{x} \wedge \mathbf{x}\cap\mathbf{y}\equiv O)$.

PROOF. The hypothesis $\mathbf{x}\subseteq\mathbf{h}$ entails $M\mathbf{x}$ by the definition of $\subseteq$. Hence $\exists\mathbf{y}M\mathbf{y}$ by lemma 3(i), whence $m\mathbf{z}U\mathbf{z}$ by axiom 2(ii). Also from $\exists\mathbf{y}M\mathbf{y}$ it follows that $V_1 = \mathbf{z}\cdot U\mathbf{z}$ by lemma 9(iii). Hence $\mathsf{E}!V_1$ by the strength of identity, whence $\mathsf{E}!V_2$ by lemma 9(vi). We consider two cases separately: (i) $\mathbf{h}=V_1$ and (ii) $\mathbf{h}\neq V_1$.

CASE (i) $\mathbf{h}=V_1$
Since by hypothesis $\mathbf{x}\subseteq\mathbf{h}$, it follows that $\mathbf{y}\in\mathbf{x}\rightarrow\mathbf{y}\in\mathbf{h}$ by the definition of $\subseteq$. Hence $\mathbf{y}\in\mathbf{x}\rightarrow\mathbf{y}\in V_1$, whence $\mathbf{y}\in\mathbf{x}\rightarrow U\mathbf{y}$ by lemma 9(ii). Hence $\mathbf{y}\in\mathbf{x}\rightarrow\neg\exists\mathbf{z}\ \mathbf{z}\in\mathbf{y}$ by the definition of U and axiom 1(iv). Since $M\mathbf{x}$, it follows by the definition of M that $m\mathbf{y}\ \mathbf{y}\in\mathbf{x}$. Hence $\exists\mathbf{y}(\mathbf{y}\in\mathbf{x} \wedge \neg\exists\mathbf{z}(\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y}))$, whence $\exists\mathbf{y}(\mathbf{y}\in\mathbf{x} \wedge \mathbf{x}\cap\mathbf{y}\equiv O)$ by theorem 2(i) and lemma 7(ix).

CASE (ii) $\mathbf{h}\neq V_1$

1 Since by hypothesis $\mathbf{x}\subseteq\mathbf{h}$, it follows that $\mathbf{y}\in\mathbf{x}\rightarrow\mathbf{y}\in\mathbf{h}$, by the definition of $\subseteq$. Since $\mathbf{h}\neq V_1$, it follows by theorem 6 that $\mathbf{y}\in\mathbf{x}\rightarrow V\mathbf{y}$. For a reductio suppose that $\neg\exists\mathbf{y}(\mathbf{y}\in\mathbf{x} \wedge \neg\exists\mathbf{z}(\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y})$, whence $\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow\exists\mathbf{z}(\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y}))$. Since $M\mathbf{x}$, it follows by the definition of M that $m\mathbf{y}\ \mathbf{y}\in\mathbf{x}$. Consider an arbitrary level $\mathbf{v}$ such that $\mathbf{v}\in\mathbf{x}$. Then $\mathbf{w}\in\mathbf{x} \wedge \mathbf{w}\in\mathbf{v}$ for some level $\mathbf{w}$. By the same reasoning applied to $\mathbf{w}$, it follows from $\mathbf{w}\in\mathbf{x}$ that $\mathbf{w_1}\in\mathbf{x} \wedge \mathbf{w_1}\in\mathbf{w}$ for some level $\mathbf{w_1}$. And similarly $\mathbf{w_2}\in\mathbf{x} \wedge \mathbf{w_2}\in\mathbf{w_1}$ for some level $\mathbf{w_2}$. Also since $\mathbf{y}\in\mathbf{x}\rightarrow\mathbf{y}\in\mathbf{h}$, it follows that $\mathbf{v}\in\mathbf{h}$ and $\mathbf{w}\in\mathbf{h}$.

2 Let b be short for $\mathbf{z}\cdot\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow\mathbf{z}\in\mathbf{y})$. We shall prove (i) $\mathsf{E}!b$, (ii) $b\subseteq\mathbf{w}$, (iii) $\mathbf{v}=\mathrm{acc}(\mathbf{h}\cap\mathbf{v})$, (iv) $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\in\mathbf{h} \wedge \mathbf{z}\in\mathbf{v})$, and (v) $\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow b\in\mathbf{y})$.

3 For (i), since $\mathbf{y}\in\mathbf{x}\rightarrow V\mathbf{y}$ (from step 1), and $V\mathbf{y}\rightarrow\forall\mathbf{z}(U\mathbf{z}\rightarrow\mathbf{z}\in\mathbf{y})$ by lemma 8(iv), it follows that $\forall\mathbf{z}(U\mathbf{z}\rightarrow\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow\mathbf{z}\in\mathbf{y}))$. Since $m\mathbf{z}U\mathbf{z}$, it follows that $m\mathbf{z}(E\mathbf{z} \wedge \forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow\mathbf{z}\in\mathbf{y}))$ by axiom 2(i), whence $\mathsf{E}!b$ by lemma 2(i).

4 For (ii), since $\mathbf{w}\in\mathbf{x}$, it follows that $\forall\mathbf{z}(\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow\mathbf{z}\in\mathbf{y})\rightarrow\mathbf{z}\in\mathbf{w})$. Since $\mathsf{E}!b$, it follows that $b = \mathbf{z}\cdot\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow\mathbf{z}\in\mathbf{y})$. Hence by lemma 4 Mb and also $\mathbf{z}\in b\rightarrow\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow\mathbf{z}\in\mathbf{y})$, whence $\mathbf{z}\in b\rightarrow\mathbf{z}\in\mathbf{w}$. Hence $b\subseteq\mathbf{w}$ by the definition of $\subseteq$.

5 For (iii) and (iv), we shall first prove (iiia) $\mathbf{v}\neq V_1$ and (iiib) $\mathbf{v}\neq V_2$.

6 For (iiia), for a reductio suppose that $\mathbf{v} = V_1 = \mathbf{z}\cdot U\mathbf{z}$. Since $\mathbf{w}\in\mathbf{v}$, it follows by lemma 9(ii) that $U\mathbf{w}$. But by lemma 8(i), $M\mathbf{w}$, whence by the definition of U, $\neg U\mathbf{w}$. Contradiction. Hence $\mathbf{v}\neq V_1$.

7 For (iiib), for a reductio suppose that $\mathbf{v}=V_2$. Since $\mathbf{w}\in\mathbf{v}$, it follows by lemma 9(v) that $U\mathbf{w} \vee \mathbf{w}\subseteq V_1$. Since $\neg U\mathbf{w}$ it follows that $\mathbf{w}\subseteq V_1$. Hence $\mathbf{w_1}\in V_1$ by the definition of $\subseteq$, whence $U\mathbf{w_1}$ by lemma 9(ii). But by lemma 8(i), $M\mathbf{w_1}$, whence $\neg U\mathbf{w_1}$ by the definition of U. Contradiction. Hence $\mathbf{v}\neq V_2$.

8 Since $\mathbf{h}\neq V_1$ and $\mathbf{v}\in\mathbf{h}$ and $\mathbf{v}\neq V_1$ and $\mathbf{v}\neq V_2$, it follows by theorems 5(i) and 5(iii) that $\mathbf{v}=\mathrm{acc}(\mathbf{h}\cap\mathbf{v})$ and $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\in\mathbf{h} \wedge \mathbf{z}\in\mathbf{v})$.

9 For (vi), since $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\in\mathbf{h} \wedge \mathbf{z}\in\mathbf{v})$, it follows by the definition of $\cap$ and lemma 2(ii) that $\mathbf{h}\cap\mathbf{x} = \mathbf{z}\cdot(\mathbf{z}\in\mathbf{h} \wedge \mathbf{z}\in\mathbf{x})$. Since $\mathbf{w}\in\mathbf{h} \wedge \mathbf{w}\in\mathbf{v}$, it follows by lemmas 4 and 5(i) that $\mathbf{w}\in\mathbf{h}\cap\mathbf{v}$. By the definition of acc, $\mathbf{v} = \mathrm{acc}(\mathbf{h}\cap\mathbf{v}) = \mathbf{z}\cdot(U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{h}\cap\mathbf{v} \wedge (\mathbf{z}\in\mathbf{y} \vee \mathbf{z}\subseteq\mathbf{y})))$. Since $b\subseteq\mathbf{w}$, it follows that Eb by theorem 1(ii) and the definition of S. Since Eb and $b\subseteq\mathbf{w}$ and $\mathbf{w}\in\mathbf{h}\cap\mathbf{v}$, it follows that $b\in\mathrm{acc}(\mathbf{h}\cap\mathbf{v})$ by lemma 4, whence $b\in\mathbf{v}$. Since $b\in\mathbf{v}$ for arbitrary $\mathbf{v}\in\mathbf{x}$, we can generalize to get $\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow b\in\mathbf{y})$.

10 We can now proceed to the reductio initiated in step 1. Since by lemma 4 $(E\mathbf{z} \wedge \forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow\mathbf{z}\in\mathbf{y}))\rightarrow\mathbf{z}\in b$ then in particular $(Eb \wedge \forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow b\in\mathbf{y})\rightarrow b\in b$. Since $(Eb \wedge \forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow b\in\mathbf{y})$, it follows that $b\in b$, contrary to axiom 1(iii). Hence $\exists\mathbf{y}(\mathbf{y}\in\mathbf{x} \wedge \neg\exists\mathbf{z}(\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\in\mathbf{y}))$, whence $\exists\mathbf{y}(\mathbf{y}\in\mathbf{x} \wedge \mathbf{x}\cap\mathbf{y}\equiv O)$ by theorem 2(i) and lemma 7(ix).

THEOREM 8. *Levels are transitive sets* *Let* $\mathbf{x}\in\mathbf{y}$ *and* $\mathbf{y}\in\mathbf{v}$, *then* $\mathbf{x}\in\mathbf{v}$.

PROOF. From the hypothesis $\mathbf{x}\in\mathbf{y}$ it follows that $E\mathbf{x} \wedge M\mathbf{y}$ by lemma 5(i), whence by the definition of U, $\neg U\mathbf{y}$. Since $M\mathbf{y}$ it follows that $\exists\mathbf{z}\, M\mathbf{z}$ by lemma 3(i), whence $V_1 = \mathbf{z}{\cdot}U\mathbf{z}$ by lemma 9(iii). Hence $\mathsf{E}!V_1$ by the strength of identity, whence $\mathsf{E}!V_2$ by lemma 9(vi). From the hypothesis $\mathbf{y}\in\mathbf{v}$ it follows that $\mathsf{E}!\mathbf{v}$ by axiom 1(i). We tackle three cases separately: (i) $\mathbf{v}=V_1$, (ii) $\mathbf{v}=V_2$ and (iii) $\mathbf{v}\neq V_1$ and $\mathbf{v}\neq V_2$.

CASE (i) $\mathbf{v}=V_1$
From the hypothesis $\mathbf{y}\in\mathbf{v}$ it follows by lemma 9(ii) that $U\mathbf{y}$. But also $\neg U\mathbf{y}$. Hence $\mathbf{x}\in\mathbf{v}$ by the tautology $A\wedge\neg A\rightarrow B$.

CASE (ii) $\mathbf{v}=V_2$
From the hypothesis $\mathbf{y}\in\mathbf{v}$ it follows by lemma 9(v) that $U\mathbf{y} \vee \mathbf{y}\subseteq V_1$. Since $\neg U\mathbf{y}$, it follows that $\mathbf{y}\subseteq V_1$. Since $\mathbf{x}\in\mathbf{y}$, it follows that $\mathbf{x}\in V_1$ by the definition of $\subseteq$. Hence $U\mathbf{x}$ by lemma 9(ii). Hence $\mathbf{x}\in V_2$ by lemma 9(v), whence $\mathbf{x}\in\mathbf{v}$.

CASE (iii) $\mathbf{v}\neq V_1$ and $\mathbf{v}\neq V_2$

1 By the definition of V, $\mathbf{v}$=acc($\mathbf{h}$) for some history $\mathbf{h}$. For a reductio suppose that $\mathbf{h}=V_1$. Then $\mathbf{v}$ = acc(V_1) = V_1 by lemma 9(i). Contradiction. Hence $\mathbf{h}\neq V_1$, whence by the definition of acc, theorem 6 and lemma 1(i), $\mathbf{v} = \mathbf{z}{\cdot}(U\mathbf{z} \vee \exists\mathbf{u}(\mathbf{u}\in\mathbf{h} \wedge (\mathbf{z}\in\mathbf{u} \vee \mathbf{z}\subseteq\mathbf{u})))$. Since $\mathbf{y}\in\mathbf{v}$, it follows by lemma 4 that $U\mathbf{y} \vee \exists\mathbf{u}(\mathbf{u}\in\mathbf{h} \wedge (\mathbf{y}\in\mathbf{u} \vee \mathbf{y}\subseteq\mathbf{u}))$. Since $\neg U\mathbf{y}$, it follows that $\mathbf{y}\in\mathbf{w} \vee \mathbf{y}\subseteq\mathbf{w}$ for some $\mathbf{w}\in\mathbf{h}$. We tackle three cases separately: (a) $\mathbf{w}=V_1$, (b) $\mathbf{w}=V_2$, and (c) $\mathbf{w}\neq V_1$ and $\mathbf{w}\neq V_2$.

2 For case (a), for a reductio suppose $\mathbf{y}\in\mathbf{w}$. Then $\mathbf{y}\in V_1$, whence by lemma 9(ii), $U\mathbf{y}$. Contradiction. Hence $\mathbf{y}\subseteq\mathbf{w}$. Since $\mathbf{x}\in\mathbf{y}$, it follows by the definition of $\subseteq$ that $\mathbf{x}\in\mathbf{w}$. Since $\mathbf{v} = \mathbf{z}{\cdot}(U\mathbf{z} \vee \exists\mathbf{u}(\mathbf{u}\in\mathbf{h} \wedge (\mathbf{z}\in\mathbf{u} \vee \mathbf{z}\subseteq\mathbf{u})))$, it follows by lemma 4 that $\mathbf{x}\in\mathbf{v}$.

3 For case (b) suppose $\mathbf{y}\in\mathbf{w}$. Then $\mathbf{y}\in V_2$, whence $U\mathbf{y} \vee \mathbf{y}\subseteq V_1$ by lemma 9(v). Since $\neg U\mathbf{y}$, it follows that $\mathbf{y}\subseteq V_1$. Since $\mathbf{x}\in\mathbf{y}$, it follows by the definition of $\subseteq$ that $\mathbf{x}\in V_1$, whence $U\mathbf{x}$ by lemma 9(ii). Hence by lemma 4 $\mathbf{x}\in\mathbf{v}$. Suppose instead that $\mathbf{y}\subseteq\mathbf{w}$. Since $\mathbf{x}\in\mathbf{y}$, it follows by the definition of $\subseteq$ that $\mathbf{x}\in\mathbf{w}$, whence $\mathbf{x}\in\mathbf{v}$ by lemma 4.

4 For case (c) we deal with two subcases separately: (ci) $\exists_1\mathbf{u}(\mathbf{u}\in\mathbf{h} \wedge (\mathbf{y}\in\mathbf{u} \vee \mathbf{y}\subseteq\mathbf{u}))$, and (cii) $m\mathbf{u}(\mathbf{u}\in\mathbf{h} \wedge (\mathbf{y}\in\mathbf{u} \vee \mathbf{y}\subseteq\mathbf{u}))$.

5 For case (ci), $\mathbf{u}\in\mathbf{h} \wedge (\mathbf{y}\in\mathbf{u} \vee \mathbf{y}\subseteq\mathbf{u})$ for some unique level $\mathbf{u}$, namely $\mathbf{w}$. Since $\mathbf{h}\neq V_1$, $\mathbf{w}\in\mathbf{h}$, $\mathbf{w}\neq V_1$ and $\mathbf{w}\neq V_2$, it follows that $\mathbf{w}$=acc($\mathbf{h}\cap\mathbf{w}$) and $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\in\mathbf{h} \wedge \mathbf{z}\in\mathbf{w})$ by theorems 5(i) and 5(iii). Hence by the definition of acc, $\mathbf{w} = \mathbf{z}{\cdot}(U\mathbf{z} \vee \exists\mathbf{z_1}(\mathbf{z_1}\in\mathbf{h}\cap\mathbf{w} \wedge (\mathbf{z}\in\mathbf{z_1} \vee \mathbf{z}\subseteq\mathbf{z_1})))$. For a reductio suppose $\mathbf{y}\in\mathbf{w}$. Then by lemma 4, $U\mathbf{y} \vee \exists\mathbf{z_1}(\mathbf{z_1}\in\mathbf{h}\cap\mathbf{w} \wedge (\mathbf{y}\in\mathbf{z_1} \vee \mathbf{y}\subseteq\mathbf{z_1}))$. Since $\neg U\mathbf{y}$, it follows that $\exists\mathbf{z_1}(\mathbf{z_1}\in\mathbf{h}\cap\mathbf{w} \wedge (\mathbf{y}\in\mathbf{z_1} \vee \mathbf{y}\subseteq\mathbf{z_1}))$. Since $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\in\mathbf{h} \wedge \mathbf{z}\in\mathbf{w})$, it follows by the definition of $\cap$ and lemma 2(ii) that $\mathbf{h}\cap\mathbf{w} = \mathbf{z}{\cdot}(\mathbf{z}\in\mathbf{h} \wedge \mathbf{z}\in\mathbf{w})$, whence by lemma 4 for some $\mathbf{z_1}$, $\mathbf{z_1}\in\mathbf{h} \wedge \mathbf{z_1}\in\mathbf{w} \wedge (\mathbf{y}\in\mathbf{z_1} \vee \mathbf{y}\subseteq\mathbf{z_1})$. Since $\mathbf{z_1}\in\mathbf{h}$ and $\mathbf{h}\neq V_1$, it follows by theorem 6 that $V\mathbf{z_1}$. Since $\mathbf{z_1}\in\mathbf{w}$, it follows by

axiom 1(iii) that $\mathbf{z}_1\neq\mathbf{w}$. But $\mathbf{z}_1\in\mathbf{h} \wedge (\mathbf{y}\in\mathbf{z}_1 \vee \mathbf{y}\subseteq\mathbf{z}_1)$ and $V\mathbf{z}_1$ and $\mathbf{z}_1\neq\mathbf{w}$ are together contrary to $\exists_1\mathbf{u}(\mathbf{u}\in\mathbf{h} \wedge (\mathbf{y}\in\mathbf{u} \vee \mathbf{y}\subseteq\mathbf{u}))$. Hence $\mathbf{y}\subseteq\mathbf{w}$. Since $\mathbf{x}\in\mathbf{y}$, it follows by the definition of $\subseteq$ that $\mathbf{x}\in\mathbf{w}$, and so by lemma 4 $\mathbf{x}\in\mathbf{v}$.

6 For case (cii) let b be short for $\mathbf{u}\cdot(\mathbf{u}\in\mathbf{h} \wedge (\mathbf{y}\in\mathbf{u} \vee \mathbf{y}\subseteq\mathbf{u}))$. Since $m\mathbf{u}(\mathbf{u}\in\mathbf{h} \wedge (\mathbf{y}\in\mathbf{u} \vee \mathbf{y}\subseteq\mathbf{u}))$, it follows that $\mathsf{E}!b$ by lemmas 2(i) and 8(ii), whence $b = \mathbf{u}\cdot(\mathbf{u}\in\mathbf{h} \wedge (\mathbf{y}\in\mathbf{u} \vee \mathbf{y}\subseteq\mathbf{u}))$. Hence by lemma 4, Mb and also $\mathbf{z}\in b\rightarrow\mathbf{z}\in\mathbf{h}$, whence by the definition of $\subseteq$, $b\subseteq\mathbf{h}$. Hence by lemma 7(ix) and theorems 2(i) and 7, $\neg\exists\mathbf{z}_2(\mathbf{z}_2\in b \wedge \mathbf{z}_2\in\mathbf{z}_1)$ for some $\mathbf{z}_1\in b$. Hence by lemma 4, $V\mathbf{z}_1 \wedge \mathbf{z}_1\in\mathbf{h} \wedge (\mathbf{y}\in\mathbf{z}_1 \vee \mathbf{y}\subseteq\mathbf{z}_1)$. We deal with three subcases separately: (ciiα) $\mathbf{z}_1=V_1$, (ciiβ) $\mathbf{z}_1=V_2$, and (ciiγ) $\mathbf{z}_1\neq V_1 \wedge \mathbf{z}_1\neq V_2$.

7 In case (ciiα), it follows that $\mathbf{x}\in\mathbf{v}$ by the reasoning in step 2.

8 In case (ciiβ), it follows that $\mathbf{x}\in\mathbf{v}$ by the reasoning in step 3.

9 In case (ciiγ), since $\mathbf{h}\neq V_1$, $\mathbf{z}_1\in\mathbf{h}$, $\mathbf{z}_1\neq V_1$ and $\mathbf{z}_1\neq V_2$, it follows that $\mathbf{z}_1=\mathrm{acc}(\mathbf{h}\cap\mathbf{z}_1)$ and $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\in\mathbf{h} \wedge \mathbf{z}\in\mathbf{z}_1)$ by theorems 5(i) and 5(iii). For a reductio suppose $\mathbf{y}\in\mathbf{z}_1$. Then $\mathbf{y}\in\mathrm{acc}(\mathbf{h}\cap\mathbf{z}_1)$, whence by the definition of acc, $\mathbf{y}\in\mathbf{z}\cdot(U\mathbf{z} \vee \exists\mathbf{z}_3(\mathbf{z}_3\in\mathbf{h}\cap\mathbf{z}_1 \wedge (\mathbf{z}\in\mathbf{z}_3 \vee \mathbf{z}\subseteq\mathbf{z}_3)))$. Since $\neg U\mathbf{y}$, it follows by lemma 4 that $\mathbf{y}\in\mathbf{z}_3 \vee \mathbf{y}\subseteq\mathbf{z}_3$ for some $\mathbf{z}_3\in\mathbf{h}\cap\mathbf{z}_1$. Since $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\in\mathbf{h} \wedge \mathbf{z}\in\mathbf{z}_1)$, it follows by the definition of $\cap$ and lemma 2(ii) that $\mathbf{h}\cap\mathbf{z}_1=\mathbf{z}\cdot(\mathbf{z}\in\mathbf{h} \wedge \mathbf{z}\in\mathbf{z}_1)$, whence by lemma 4 $\mathbf{z}_3\in\mathbf{h} \wedge \mathbf{z}_3\in\mathbf{z}_1$. Since $\mathbf{h}\neq V_1$ and $\mathbf{z}_3\in\mathbf{h}$, it follows by theorem 6 that $V\mathbf{z}_3$. Since $V\mathbf{z}_3$ and $\mathbf{z}_3\in\mathbf{h} \wedge (\mathbf{y}\in\mathbf{z}_3 \vee \mathbf{y}\subseteq\mathbf{z}_3)$, it follows by lemmas 4 and 5(i) that $\mathbf{z}_3\in b$. But $\mathbf{z}_3\in b$ and $\mathbf{z}_3\in\mathbf{z}_1$ are together contrary to $\neg\exists\mathbf{z}_2(\mathbf{z}_2\in b \wedge \mathbf{z}_2\in\mathbf{z}_1)$. Hence $\mathbf{y}\notin\mathbf{z}_1$, whence $\mathbf{y}\subseteq\mathbf{z}_1$. Since $\mathbf{x}\in\mathbf{y}$, it follows by the definition of $\subseteq$ that $\mathbf{x}\in\mathbf{z}_1$, whence $\mathbf{x}\in\mathbf{v}$ by lemma 4.

COROLLARY. *Let* $M\mathbf{y}$ *and* $\mathbf{y}\in\mathbf{v}$, *then* $\mathbf{y}\subseteq\mathbf{v}$.

PROOF. Since $\mathbf{y}\in\mathbf{v}$, it follows that $\mathbf{x}\in\mathbf{y}\rightarrow\mathbf{x}\in\mathbf{v}$ by theorem 8, which together with $M\mathbf{y}$ entails $\mathbf{y}\subseteq\mathbf{v}$ by the definition of $\subseteq$.

THEOREM 9. *Levels are hereditary sets* *Let* $\mathbf{x}\subseteq\mathbf{y}$ *and* $\mathbf{y}\in\mathbf{v}$, *then* $\mathbf{x}\in\mathbf{v}$.

PROOF. From the hypothesis $\mathbf{x}\subseteq\mathbf{y}$ it follows that $M\mathbf{y}$ by lemma 6(i), whence by the definition of U, $\neg U\mathbf{y}$. Since $M\mathbf{y}$ it follows that $\exists\mathbf{z}\, M\mathbf{z}$ by lemma 3(i), whence $V_1 = \mathbf{z}\cdot U\mathbf{z}$ by lemma 9(iii). Hence $\mathsf{E}!V_1$ by the strength of identity, whence $\mathsf{E}!V_2$ by lemma 9(vi). From the hypothesis $\mathbf{y}\in\mathbf{v}$ it follows that $E\mathbf{y}$ by lemma 5(i), whence $S\mathbf{y}$ by the definition of S. Hence $S\mathbf{x}$ by axiom 2(iii), whence $E\mathbf{x}$ by the definition of S. Also from $\mathbf{y}\in\mathbf{v}$ it follows that $\mathsf{E}!\mathbf{v}$ by axiom 1(i). We tackle three cases separately: (i) $\mathbf{v}=V_1$, (ii) $\mathbf{v}=V_2$ and (iii) $\mathbf{v}\neq V_1$ and $\mathbf{v}\neq V_2$.

CASE (i) $\mathbf{v}=V_1$

From the hypothesis $\mathbf{y}\in\mathbf{v}$ it follows by lemma 9(ii) that $U\mathbf{y}$. But also $\neg U\mathbf{y}$. Hence $\mathbf{x}\in\mathbf{v}$ by the tautology $A\wedge\neg A\rightarrow B$.

CASE (ii) $\mathbf{v}=V_2$

From the hypothesis $\mathbf{y}\in\mathbf{v}$ it follows by lemma 9(v) that $U\mathbf{y} \vee \mathbf{y}\subseteq V_1$. Since $\neg U\mathbf{y}$, it follows that $\mathbf{y}\subseteq V_1$. Since $\mathbf{x}\subseteq\mathbf{y}$, it follows that $\mathbf{x}\subseteq V_1$ by the definition of $\subseteq$. Hence $\mathbf{x}\in V_2$ by lemma 9(v), whence $\mathbf{x}\in\mathbf{v}$.

CASE (iii) $\mathbf{v}\neq V_1$ and $\mathbf{v}\neq V_2$.

1 By the definition of V, $\mathbf{v}=\mathrm{acc}(\mathbf{h})$ for some history $\mathbf{h}$. For a reductio suppose that $\mathbf{h}=V_1$. Then $\mathbf{v}=\mathrm{acc}(V_1)=V_1$ by lemma 9(i). Contradiction. Hence $\mathbf{h}\neq V_1$, whence by the definition of acc, theorem 6 and lemma 1(i), $\mathbf{v}=\mathbf{z}\cdot(U\mathbf{z}\vee\exists\mathbf{u}(\mathbf{u}\in\mathbf{h}\wedge(\mathbf{z}\in\mathbf{u}\vee\mathbf{z}\subseteq\mathbf{u})))$. Since $\mathbf{y}\in\mathbf{v}$, it follows by lemma 4 that $U\mathbf{y}\vee\exists\mathbf{u}(\mathbf{u}\in\mathbf{h}\wedge(\mathbf{y}\in\mathbf{u}\vee\mathbf{y}\subseteq\mathbf{u}))$. Since $\neg U\mathbf{y}$, it follows that $\mathbf{y}\in\mathbf{w}\vee\mathbf{y}\subseteq\mathbf{w}$ for some $\mathbf{w}\in\mathbf{h}$. We tackle three cases separately: (a) $\mathbf{w}=V_1$, (b) $\mathbf{w}=V_2$, and (c) $\mathbf{w}\neq V_1$ and $\mathbf{w}\neq V_2$.

2 For case (a), for a reductio suppose $\mathbf{y}\in\mathbf{w}$. Then $\mathbf{y}\in V_1$, whence by lemma 9(ii), $U\mathbf{y}$. Contradiction. Hence $\mathbf{y}\subseteq\mathbf{w}$. Since $\mathbf{x}\subseteq\mathbf{y}$, it follows by the definition of $\subseteq$ that $\mathbf{x}\subseteq\mathbf{w}$. Since $\mathbf{v}=\mathbf{z}\cdot(U\mathbf{z}\vee\exists\mathbf{u}(\mathbf{u}\in\mathbf{h}\wedge(\mathbf{z}\in\mathbf{u}\vee\mathbf{z}\subseteq\mathbf{u})))$, it follows by lemma 4 that $\mathbf{x}\in\mathbf{v}$.

3 For case (b) suppose $\mathbf{y}\in\mathbf{w}$. Then $\mathbf{y}\in V_2$, whence $U\mathbf{y}\vee\mathbf{y}\subseteq V_1$ by lemma 9(v). Since $\neg U\mathbf{y}$, it follows that $\mathbf{y}\subseteq V_1$. Since $\mathbf{x}\subseteq\mathbf{y}$, it follows by the definition of $\subseteq$ that $\mathbf{x}\subseteq V_1$. Hence by lemma 9(v), $\mathbf{x}\in V_2$, whence $\mathbf{x}\in\mathbf{w}$. So by lemma 4 $\mathbf{x}\in\mathbf{v}$. Suppose instead that $\mathbf{y}\subseteq\mathbf{w}$. Since $\mathbf{x}\subseteq\mathbf{y}$, it follows by the definition of $\subseteq$ that $\mathbf{x}\subseteq\mathbf{w}$, whence $\mathbf{x}\in\mathbf{v}$ by lemma 4.

4 For case (c) we deal with two subcases separately: (ci) $\exists_1\mathbf{u}(\mathbf{u}\in\mathbf{h}\wedge(\mathbf{y}\in\mathbf{u}\vee\mathbf{y}\subseteq\mathbf{u}))$, and (cii) $m\mathbf{u}(\mathbf{u}\in\mathbf{h}\wedge(\mathbf{y}\in\mathbf{u}\vee\mathbf{y}\subseteq\mathbf{u}))$.

5 For case (ci), $\mathbf{u}\in\mathbf{h}\wedge(\mathbf{y}\in\mathbf{u}\vee\mathbf{y}\subseteq\mathbf{u})$ for some unique level $\mathbf{u}$, namely $\mathbf{w}$. Since $\mathbf{h}\neq V_1$, $\mathbf{w}\in\mathbf{h}$, $\mathbf{w}\neq V_1$ and $\mathbf{w}\neq V_2$, it follows that $\mathbf{w}=\mathrm{acc}(\mathbf{h}\cap\mathbf{w})$ and $m\mathbf{z}(E\mathbf{z}\wedge\mathbf{z}\in\mathbf{h}\wedge\mathbf{z}\in\mathbf{w})$ by theorems 5(i) and 5(iii). Hence by the definition of acc, $\mathbf{w}=\mathbf{z}\cdot(U\mathbf{z}\vee\exists\mathbf{z_1}(\mathbf{z_1}\in\mathbf{h}\cap\mathbf{w}\wedge(\mathbf{z}\in\mathbf{z_1}\vee\mathbf{z}\subseteq\mathbf{z_1})))$. For a reductio suppose $\mathbf{y}\in\mathbf{w}$. Then by lemma 4, $U\mathbf{y}\vee\exists\mathbf{z_1}(\mathbf{z_1}\in\mathbf{h}\cap\mathbf{w}\wedge(\mathbf{y}\in\mathbf{z_1}\vee\mathbf{y}\subseteq\mathbf{z_1}))$. Since $\neg U\mathbf{y}$, it follows that $\exists\mathbf{z_1}(\mathbf{z_1}\in\mathbf{h}\cap\mathbf{w}\wedge(\mathbf{y}\in\mathbf{z_1}\vee\mathbf{y}\subseteq\mathbf{z_1}))$. Since $m\mathbf{z}(E\mathbf{z}\wedge\mathbf{z}\in\mathbf{h}\wedge\mathbf{z}\in\mathbf{w})$, it follows by the definition of $\cap$ and lemma 2(ii) that $\mathbf{h}\cap\mathbf{w}=\mathbf{z}\cdot(\mathbf{z}\in\mathbf{h}\wedge\mathbf{z}\in\mathbf{w})$, whence by lemma 4 for some $\mathbf{z_1}$, $\mathbf{z_1}\in\mathbf{h}\wedge\mathbf{z_1}\in\mathbf{w}\wedge(\mathbf{y}\in\mathbf{z_1}\vee\mathbf{y}\subseteq\mathbf{z_1})$. Since $\mathbf{z_1}\in\mathbf{h}$ and $\mathbf{h}\neq V_1$, it follows by theorem 6 that $V\mathbf{z_1}$. Since $\mathbf{z_1}\in\mathbf{w}$, it follows by axiom 1(iii) that $\mathbf{z_1}\neq\mathbf{w}$. But $\mathbf{z_1}\in\mathbf{h}\wedge(\mathbf{y}\in\mathbf{z_1}\vee\mathbf{y}\subseteq\mathbf{z_1})$ and $V\mathbf{z_1}$ and $\mathbf{z_1}\neq\mathbf{w}$ are together contrary to $\exists_1\mathbf{u}(\mathbf{u}\in\mathbf{h}\wedge(\mathbf{y}\in\mathbf{u}\vee\mathbf{y}\subseteq\mathbf{u}))$. Hence $\mathbf{y}\subseteq\mathbf{w}$. Since $\mathbf{x}\subseteq\mathbf{y}$, it follows by the definition of $\subseteq$ that $\mathbf{x}\subseteq\mathbf{w}$, and so by lemma 4 $\mathbf{x}\in\mathbf{v}$.

6 For case (cii) let b be short for $\mathbf{u}\cdot(\mathbf{u}\in\mathbf{h}\wedge(\mathbf{y}\in\mathbf{u}\vee\mathbf{y}\subseteq\mathbf{u}))$. Since $m\mathbf{u}(\mathbf{u}\in\mathbf{h}\wedge(\mathbf{y}\in\mathbf{u}\vee\mathbf{y}\subseteq\mathbf{u}))$, it follows that $\mathsf{E}!b$ by lemmas 2(i) and 8(ii), whence $b=\mathbf{u}\cdot(\mathbf{u}\in\mathbf{h}\wedge(\mathbf{y}\in\mathbf{u}\vee\mathbf{y}\subseteq\mathbf{u}))$. Hence by lemma 4, Mb and also $\mathbf{z}\in b\rightarrow\mathbf{z}\in\mathbf{h}$, whence by the definition of $\subseteq$, $b\subseteq\mathbf{h}$. Hence by lemma 7(ix), and theorems 2(i) and 7, $\neg\exists\mathbf{z_2}(\mathbf{z_2}\in b\wedge\mathbf{z_2}\in\mathbf{z_1})$ for some $\mathbf{z_1}\in b$. Hence by lemma 4, $V\mathbf{z_1}\wedge\mathbf{z_1}\in\mathbf{h}\wedge(\mathbf{y}\in\mathbf{z_1}\vee\mathbf{y}\subseteq\mathbf{z_1})$. We deal with three subcases separately: (ciiα) $\mathbf{z_1}=V_1$, (ciiβ) $\mathbf{z_1}=V_2$, and (ciiγ) $\mathbf{z_1}\neq V_1\wedge\mathbf{z_1}\neq V_2$.

7 In case (ciiα), it follows that $\mathbf{x}\in\mathbf{v}$ by the reasoning in step 2.

8 In case (ciiβ), it follows that $\mathbf{x}\in\mathbf{v}$ by the reasoning in step 3.

9 In case (ciiγ), since $\mathbf{h}\neq V_1$, $\mathbf{z_1}\in\mathbf{h}$, $\mathbf{z_1}\neq V_1$ and $\mathbf{z_1}\neq V_2$, it follows that $\mathbf{z_1}=\mathrm{acc}(\mathbf{h}\cap\mathbf{z_1})$ and $m\mathbf{z}(E\mathbf{z}\wedge\mathbf{z}\in\mathbf{h}\wedge\mathbf{z}\in\mathbf{z_1})$ by theorems 5(i) and 5(iii). For a reductio suppose $\mathbf{y}\in\mathbf{z_1}$. Then $\mathbf{y}\in\mathrm{acc}(\mathbf{h}\cap\mathbf{z_1})$, whence by the definition of acc, $\mathbf{y}\in\mathbf{z}\cdot(U\mathbf{z}\vee\exists\mathbf{z_3}(\mathbf{z_3}\in\mathbf{h}\cap\mathbf{z_1}\wedge(\mathbf{z}\in\mathbf{z_3}\vee\mathbf{z}\subseteq\mathbf{z_3})))$. Since $\neg U\mathbf{y}$, it follows by lemma 4 that $\mathbf{y}\in\mathbf{z_3}\vee\mathbf{y}\subseteq\mathbf{z_3}$ for some $\mathbf{z_3}\in\mathbf{h}\cap\mathbf{z_1}$. Since $m\mathbf{z}(E\mathbf{z}\wedge\mathbf{z}\in\mathbf{h}\wedge\mathbf{z}\in\mathbf{z_1})$, it follows by the definition of $\cap$ and lemma 2(ii) that $\mathbf{h}\cap\mathbf{z_1}=\mathbf{z}\cdot(\mathbf{z}\in\mathbf{h}\wedge\mathbf{z}\in\mathbf{z_1})$, whence by lemma 4 $\mathbf{z_3}\in\mathbf{h}\wedge\mathbf{z_3}\in\mathbf{z_1}$. Since $\mathbf{h}\neq V_1$ and $\mathbf{z_3}\in\mathbf{h}$, it follows by theorem 6 that $V\mathbf{z_3}$. Since $V\mathbf{z_3}$ and $\mathbf{z_3}\in\mathbf{h}\wedge(\mathbf{y}\in\mathbf{z_3}\vee\mathbf{y}\subseteq\mathbf{z_3})$, it follows by lemmas 4 and 5(i) that $\mathbf{z_3}\in b$. But $\mathbf{z_3}\in b$ and $\mathbf{z_3}\in\mathbf{z_1}$ are together contrary to $\neg\exists\mathbf{z_2}(\mathbf{z_2}\in b\wedge\mathbf{z_2}\in\mathbf{z_1})$. Hence $\mathbf{y}\notin\mathbf{z_1}$, whence $\mathbf{y}\subseteq\mathbf{z_1}$. Since $\mathbf{x}\subseteq\mathbf{y}$, it follows by the definition of $\subseteq$ that $\mathbf{x}\subseteq\mathbf{z_1}$, whence $\mathbf{x}\in\mathbf{v}$ by lemma 4.

THEOREM 10. *Lower levels I*

(i) $\neg\exists\mathbf{w}\ \mathbf{w}\in V_1$ *and* $\mathbf{w}{:}\mathbf{w}\in V_1 \equiv O$

(ii) *Let* $\mathsf{E}!V_2$, *then* $\exists_1\mathbf{w}\ \mathbf{w}\in V_2$ *and* $\mathbf{w}{:}\mathbf{w}\in V_2 = \iota\mathbf{w}(\mathbf{w}\in V_2) = V_1$.

(iii) *Let* $\mathsf{E}!\mathbf{v}$ *and* $\mathbf{v}\neq V_1$ *and* $\mathbf{v}\neq V_2$, *then* $m\mathbf{w}\ \mathbf{w}\in\mathbf{v}$ *and* $\mathbf{w}{:}\mathbf{w}\in\mathbf{v} = \mathbf{w}\cdot\mathbf{w}\in\mathbf{v}$.

PROOF OF (i). For a reductio suppose that $\mathbf{w}\in V_1$ for some level $\mathbf{w}$. Then by lemma 8(i), Mw, and by axiom 1(i), $\mathsf{E}!V_1$. Hence by lemma 9(ii), $U\mathbf{w}$, whence $\neg M\mathbf{w}$ by the definition of U. Contradiction. Hence $\neg\exists\mathbf{w}\ \mathbf{w}\in V_1$, whence by lemmas 2(iv) and 7(ix), $\mathbf{w}{:}\mathbf{w}\in V_1 \equiv O$.

PROOF OF (ii).

1 It follows from the hypothesis that $V_2 = \mathbf{z}\cdot(U\mathbf{z} \vee \mathbf{z}\subseteq V_1)$ by lemma 9(iv), and that $\mathsf{E}!V_1$ by lemma 9(vi). Hence by lemma 9(i), $V_1 = \mathbf{z}\cdot U\mathbf{z}$. By lemma 9(vii), $V_1\in V_2$, and by the definition of V, $V(V_1)$.

2 For a reductio suppose $\mathbf{w}\neq V_1 \wedge \mathbf{w}\in V_2$ for some level $\mathbf{w}$. Then by axiom 1(iii), $\mathbf{w}\neq V_2$. Also by lemma 9(v), $U\mathbf{w} \vee \mathbf{w}\subseteq V_1$. By lemma 8(i) $M\mathbf{w}$, whence by the definition of U, $\neg U\mathbf{w}$. Hence $\mathbf{w}\subseteq V_1$. Since $\mathbf{w}\neq V_1$, it follows by the definition of $\subset$ that $\mathbf{w}\subset V_1$, whence $\mathbf{w}\subset\mathbf{z}\cdot U\mathbf{z}$. Hence by lemma 6(iii), $\mathbf{x}\notin\mathbf{w} \wedge \mathbf{x}\in\mathbf{z}\cdot U\mathbf{z}$ for some $\mathbf{x}$, whence by lemma 4, $\mathbf{x}\notin\mathbf{w} \wedge U\mathbf{x}$, contrary to lemma 8(iv). Hence $\neg\exists\mathbf{w}(\mathbf{w}\neq V_1 \wedge \mathbf{w}\in V_2)$.

3 Since $V(V_1)$ and $V_1\in V_2$ and $\neg\exists\mathbf{w}(\mathbf{w}\neq V_1 \wedge \mathbf{w}\in V_2)$, it follows that $\exists_1\mathbf{w}\ \mathbf{w}\in V_2$ and $\iota\mathbf{w}(\mathbf{w}\in V_2) = V_1$. Since by lemma 8(ii), $V\mathbf{x}\rightarrow E\mathbf{x}$, it follows that $\exists_1\mathbf{w}(E\mathbf{w} \wedge \mathbf{w}\in V_2)$ and $\iota\mathbf{w}(\mathbf{w}\in V_2) = \iota\mathbf{w}(E\mathbf{w} \wedge \mathbf{w}\in V_2)$, whence $\mathbf{w}{:}\mathbf{w}\in V_2 = \iota\mathbf{w}(\mathbf{w}\in V_2) = V_1$ by lemma 2(iii).

PROOF OF (iii).

1 It follows from the hypothesis and the definition of V that $\mathbf{v}=\mathrm{acc}(\mathbf{h})$ for some history $\mathbf{h}$. Hence by theorem 3(ii), $M\mathbf{h} \wedge \mathbf{h}\neq V_1$, whence $m\mathbf{y}\ \mathbf{y}\in\mathbf{h}$ by the definition of M. Hence $m\mathbf{w}\ \mathbf{w}\in\mathbf{h}$ by theorem 6.

2 Consider an arbitrary level $\mathbf{u}\in\mathbf{h}$. Then $\mathbf{u}\subseteq\mathbf{u}$ by lemmas 6(ii) and 8(i). By the definition of acc, $\mathbf{v} = \mathrm{acc}(\mathbf{h}) = \mathbf{z}\cdot(U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{h} \wedge (\mathbf{z}\in\mathbf{y} \vee \mathbf{z}\subseteq\mathbf{y})))$, whence $\mathbf{u}\in\mathbf{v}$ by lemmas 4 and 8(ii). Hence $m\mathbf{w}\ \mathbf{w}\in\mathbf{v}$, whence by lemma 8(ii) $m\mathbf{w}(E\mathbf{w} \wedge \mathbf{w}\in\mathbf{v})$. Hence by lemma 2(ii), $\mathbf{w}{:}\mathbf{w}\in\mathbf{v} = \mathbf{w}\cdot\mathbf{w}\in\mathbf{v}$.

THEOREM 11. *Lower levels II*

(i) *Let* $\mathbf{v}=V_1$ *or* $\mathbf{v}=V_2$, *then* $\mathbf{v}=\mathrm{accum}(\mathbf{w}{:}\mathbf{w}\in\mathbf{v})$.

(ii) *Let* $\mathsf{E}!\mathbf{v}$ *and* $\mathbf{v}\neq V_1$ *and* $\mathbf{v}\neq V_2$, *then* $\mathbf{v}=\mathrm{acc}(\mathbf{w}{:}\mathbf{w}\in\mathbf{v})$.

PROOF OF (i). Suppose $\mathbf{v}=V_1$. Then by theorem 10(i) $\mathbf{w}{:}\mathbf{w}\in\mathbf{v} \equiv O$, whence $\mathbf{v} = \mathrm{accum}(\mathbf{w}{:}\mathbf{w}\in\mathbf{v})$ by the definition of V_1. Suppose instead that $\mathbf{v}=V_2$. Then $\mathsf{E}!V_2$ by the strength of identity, whence by theorem 10(ii) $\mathbf{w}{:}\mathbf{w}\in\mathbf{v} = V_1$. Hence $\mathbf{v}=\mathrm{accum}(\mathbf{w}{:}\mathbf{w}\in\mathbf{v})$ by the definition of V_2.

PROOF OF (ii).

1 Since by hypothesis $\mathsf{E}!\mathbf{v}$ and $\mathbf{v}\neq V_1$ and $\mathbf{v}\neq V_2$, it follows that $\mathbf{w}{:}\mathbf{w}\in\mathbf{v} = \mathbf{w}\cdot\mathbf{w}\in\mathbf{v}$ by theorem 10(iii), whence by lemma 4 $\mathbf{y}\in\mathbf{w}\cdot\mathbf{w}\in\mathbf{v}\leftrightarrow(E\mathbf{y} \wedge \mathbf{y}\in\mathbf{v} \wedge V\mathbf{y})$. It also follows from the hypothesis by the definition of V that $\mathbf{v}=\mathrm{acc}(\mathbf{h})$ for some history $\mathbf{h}$. By the

definition of acc, $\mathbf{v} = \text{acc}(\mathbf{h}) = \mathbf{z}\cdot(U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{h} \wedge (\mathbf{z}\in\mathbf{y} \vee \mathbf{z}\subseteq\mathbf{y})))$. Hence by lemma 4 $M\mathbf{v}$ and also $\mathbf{z}\in\mathbf{v}\leftrightarrow(E\mathbf{z} \wedge (U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{h} \wedge (\mathbf{z}\in\mathbf{y} \vee \mathbf{z}\subseteq\mathbf{y}))))$.

2 Since $\mathsf{E}!\mathbf{v}$ and $M\mathbf{v}$, it follows that $m\mathbf{z}(E\mathbf{z} \wedge U\mathbf{z})$ by axioms 2(i) and 2(ii), a fortiori $m\mathbf{z}(E\mathbf{z} \wedge (U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{w}\cdot\mathbf{w}\in\mathbf{v} \wedge (\mathbf{z}\in\mathbf{y} \vee \mathbf{z}\subseteq\mathbf{y}))))$. Hence $\mathsf{E}!\mathbf{z}\cdot(U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{w}\cdot\mathbf{w}\in\mathbf{v} \wedge (\mathbf{z}\in\mathbf{y} \vee \mathbf{z}\subseteq\mathbf{y})))$ by lemma 2(i), whence $\text{acc}(\mathbf{w}\cdot\mathbf{w}\in\mathbf{v}) = \mathbf{z}\cdot(U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{w}\cdot\mathbf{w}\in\mathbf{v} \wedge (\mathbf{z}\in\mathbf{y} \vee \mathbf{z}\subseteq\mathbf{y})))$ by the definition of acc. Hence by lemma 4, $M(\text{acc}(\mathbf{w}\cdot\mathbf{w}\in\mathbf{v}))$ and also $\mathbf{z}\in\text{acc}(\mathbf{w}\cdot\mathbf{w}\in\mathbf{v})\leftrightarrow(E\mathbf{z} \wedge (U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{w}\cdot\mathbf{w}\in\mathbf{v} \wedge (\mathbf{z}\in\mathbf{y} \vee \mathbf{z}\subseteq\mathbf{y}))))$. We shall prove $\mathbf{z}\in\mathbf{v}\leftrightarrow\mathbf{z}\in\text{acc}(\mathbf{w}\cdot\mathbf{w}\in\mathbf{v})$.

3 For the $\rightarrow$ half, suppose $\mathbf{z}\in\mathbf{v}$. Then $E\mathbf{z} \wedge (U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{h} \wedge (\mathbf{z}\in\mathbf{y} \vee \mathbf{z}\subseteq\mathbf{y})))$. By lemma 5(i), and theorems 3(i), 3(ii) and 6, $\mathbf{y}\in\mathbf{h}\rightarrow(E\mathbf{y} \wedge \mathbf{y}\in\mathbf{v} \wedge V\mathbf{y})$. Hence $\mathbf{y}\in\mathbf{h}\rightarrow\mathbf{y}\in\mathbf{w}\cdot\mathbf{w}\in\mathbf{v}$, whence $E\mathbf{z} \wedge (U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{w}\cdot\mathbf{w}\in\mathbf{v} \wedge (\mathbf{z}\in\mathbf{y} \vee \mathbf{z}\subseteq\mathbf{y})))$. Hence $\mathbf{z}\in\text{acc}(\mathbf{w}\cdot\mathbf{w}\in\mathbf{v})$.

4 For the $\leftarrow$ half, suppose $\mathbf{z}\in\text{acc}(\mathbf{w}\cdot\mathbf{w}\in\mathbf{v})$. Then $E\mathbf{z} \wedge (U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{w}\cdot\mathbf{w}\in\mathbf{v} \wedge (\mathbf{z}\in\mathbf{y} \vee \mathbf{z}\subseteq\mathbf{y})))$. We consider the three possibilities for $\mathbf{z}$ and deduce $\mathbf{z}\in\mathbf{v}$ in each case. First, suppose $U\mathbf{z}$. Then by lemma 8(iv) $\mathbf{z}\in\mathbf{v}$. Second, suppose $\mathbf{z}\in\mathbf{y}$ for some $\mathbf{y}\in\mathbf{w}\cdot\mathbf{w}\in\mathbf{v}$. Then $\mathbf{y}\in\mathbf{v}$, whence $\mathbf{z}\in\mathbf{v}$ by theorem 8. Third, suppose $\mathbf{z}\subseteq\mathbf{y}$ for some $\mathbf{y}\in\mathbf{w}\cdot\mathbf{w}\in\mathbf{v}$. Then $\mathbf{y}\in\mathbf{v}$, whence $\mathbf{z}\in\mathbf{v}$ by theorem 9.

5 Since $M\mathbf{v}$ and $M(\text{acc}(\mathbf{w}\cdot\mathbf{w}\in\mathbf{v}))$ and $\mathbf{z}\in\mathbf{v}\leftrightarrow\mathbf{z}\in\text{acc}(\mathbf{w}\cdot\mathbf{w}\in\mathbf{v})$, then by axiom 1(ii), $\mathbf{v} = \text{acc}(\mathbf{w}\cdot\mathbf{w}\in\mathbf{v})$, whence $\mathbf{v}=\text{acc}(\mathbf{w}:\mathbf{w}\in\mathbf{v})$.

THEOREM 12. *Lower levels III* $H(\mathbf{w}:\mathbf{w}\in\mathbf{v})$

PROOF. By lemma 8(i), $M\mathbf{v}$, whence $\mathsf{E}!\mathbf{v}$ by lemma 3(i). Hence $V_1 = \mathbf{z}\cdot U\mathbf{z}$ by lemma 9(iii), whence $\mathsf{E}!V_1$. Also SV_1 by lemma 8(iii) and the definition of V. We tackle three cases separately: (i) $\mathbf{v}=V_1$, (ii) $\mathbf{v}=V_2$, and (iii) $\mathbf{v}\neq V_1$ and $\mathbf{v}\neq V_2$.

CASE (i) $\mathbf{v}=V_1$
By theorem 10(i), $\mathbf{w}:\mathbf{w}\in\mathbf{v} \equiv O$, whence $H(\mathbf{w}:\mathbf{w}\in\mathbf{v})$ by lemma 7(viii).

CASE (ii) $\mathbf{v}=V_2$
By theorem 10(ii), $\mathbf{w}:\mathbf{w}\in\mathbf{v} = V_1$, which together with SV_1 entails $H(\mathbf{w}:\mathbf{w}\in\mathbf{v})$ by the definition of H.

CASE (iii) $\mathbf{v}\neq V_1$ and $\mathbf{v}\neq V_2$

1 By theorem 10(iii), $m\mathbf{w}\ \mathbf{w}\in\mathbf{v}$. Since $\forall\mathbf{w}(\mathbf{w}\in\mathbf{v}\rightarrow\mathbf{w}\in\mathbf{v})$, it follows that $S(\mathbf{w}\cdot\mathbf{w}\in\mathbf{v})$ by theorem 1(iii) and lemma 8(ii). It also follows by theorem 10(iii) that $\mathbf{w}:\mathbf{w}\in\mathbf{v} = \mathbf{w}\cdot\mathbf{w}\in\mathbf{v}$, whence by lemmas 4 and 8(ii) $m\mathbf{w_1}\ \mathbf{w_1}\in\mathbf{w}:\mathbf{w}\in\mathbf{v}$. Consider an arbitrary level $\mathbf{u}\in\mathbf{w}:\mathbf{w}\in\mathbf{v}$. We shall prove that (a) $\mathbf{u}=V_1 \vee \mathbf{u}=V_2 \rightarrow \mathbf{u}=\text{accum}((\mathbf{w}:\mathbf{w}\in\mathbf{v})\cap\mathbf{u})$, and (b) $\mathbf{u}\neq V_1 \wedge \mathbf{u}\neq V_2 \rightarrow \mathbf{u}=\text{acc}((\mathbf{w}:\mathbf{w}\in\mathbf{v})\cap\mathbf{u})$.

2 For (a) we prove: (ai) $\mathbf{u}=V_1 \rightarrow \mathbf{u}=\text{accum}((\mathbf{w}:\mathbf{w}\in\mathbf{v})\cap\mathbf{u})$, and (aii) $\mathbf{u}=V_2 \rightarrow \mathbf{u}=\text{accum}((\mathbf{w}:\mathbf{w}\in\mathbf{v})\cap\mathbf{u})$.

3 For (ai) suppose $\mathbf{u}=V_1$. Then $\mathbf{u}=\text{accum}(O)$ by the definition of V_1. By lemmas 4 and 8(i), and the definition of U, $\mathbf{x}\in\mathbf{w}:\mathbf{w}\in\mathbf{v}\rightarrow \neg U\mathbf{x}$. Also by lemma 9(ii), $\mathbf{x}\in\mathbf{u}\rightarrow U\mathbf{x}$. Hence $\neg\exists\mathbf{x}(\mathbf{x}\in\mathbf{w}:\mathbf{w}\in\mathbf{v} \wedge \mathbf{x}\in\mathbf{u})$, whence $(\mathbf{x}: \mathbf{x}\in\mathbf{w}:\mathbf{w}\in\mathbf{v} \wedge \mathbf{x}\in\mathbf{u})\equiv O$ by lemmas 2(iv) and 7(ix). Hence $(\mathbf{w}:\mathbf{w}\in\mathbf{v})\cap\mathbf{u}\equiv O$ by the definition of $\cap$, whence $u=\text{accum}((\mathbf{w}:\mathbf{w}\in\mathbf{v})\cap\mathbf{u})$.

4 For (aii) suppose $\mathbf{u}=V_2$. Then $\mathbf{u}=\text{accum}(V_1)$ by the definition of V_2. Since $\mathbf{v}\neq V_1$ and $\mathbf{v}\neq V_2$, it follows by the definition of V that $\mathbf{v}=\text{acc}(\mathbf{h})$ for some history $\mathbf{h}$. By theorem 3(ii) $M\mathbf{h} \wedge \mathbf{h}\neq V_1$, whence $m\mathbf{y}\ \mathbf{y}\in\mathbf{h}$ by the definition of M. Hence $\mathbf{w_1}\in\mathbf{h}$ for some level $\mathbf{w_1}$ by theorem 6. By lemma 8(iv), $U\mathbf{z}\rightarrow\mathbf{z}\in\mathbf{w_1}$, whence by lemma 9(ii), $\mathbf{z}\in V_1\rightarrow\mathbf{z}\in\mathbf{w_1}$. Since SV_1, it follows that $MV_1 \wedge EV_1$ by the definition of S. Hence $V_1\subseteq\mathbf{w_1}$ by the definition of $\subseteq$. By the definition of acc and lemma 4, $\mathbf{z}\in\mathbf{v}\leftrightarrow(E\mathbf{z} \wedge (U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{h} \wedge (\mathbf{z}\in\mathbf{y} \vee \mathbf{z}\subseteq\mathbf{y}))))$. Hence $V_1\in\mathbf{v}$, whence $V_1\in\mathbf{w}{:}\mathbf{w}\in\mathbf{v}$ by lemma 4 and the definition of V. Since by hypothesis $\mathbf{u}=V_2$, it follows that $V_1 = \iota\mathbf{w_1}(\mathbf{w_1}\in\mathbf{u})$ by theorem 10(ii). Since $V_1\in\mathbf{w}{:}\mathbf{w}\in\mathbf{v}$, it follows that $V_1 = \iota\mathbf{w_1}(\mathbf{w_1}\in\mathbf{w}{:}\mathbf{w}\in\mathbf{v} \wedge \mathbf{w_1}\in\mathbf{u})$. Hence $\exists_1\mathbf{x}(\mathbf{x}\in\mathbf{w}{:}\mathbf{w}\in\mathbf{v} \wedge \mathbf{x}\in\mathbf{u})$ by lemma 4, whence $(\mathbf{w}{:}\mathbf{w}\in\mathbf{v})\cap\mathbf{u} = V_1$ by lemmas 2(iii) and 8(ii), and the definition of $\cap$. Hence $\mathbf{u}=\text{accum}((\mathbf{w}{:}\mathbf{w}\in\mathbf{v})\cap\mathbf{u})$.

5 From (ai) and (aii) it follows that $\mathbf{u}=V_1 \vee \mathbf{u}=V_2 \rightarrow \mathbf{u}=\text{accum}((\mathbf{w}{:}\mathbf{w}\in\mathbf{v})\cap\mathbf{u})$.

6 For (b), suppose $\mathbf{u}\neq V_1 \wedge \mathbf{u}\neq V_2$. Since $\mathbf{u}\in\mathbf{w}{:}\mathbf{w}\in\mathbf{v}$, it follows by lemma 4 that $\mathbf{u}\in\mathbf{v}$. Hence by theorem 8, $\mathbf{w_1}\in\mathbf{u}\rightarrow\mathbf{w_1}\in\mathbf{v}$, whence $(\mathbf{w_1}\in\mathbf{v} \wedge \mathbf{w_1}\in\mathbf{u})\leftrightarrow\mathbf{w_1}\in\mathbf{u}$. Hence by lemmas 4 and 8(ii), $(\mathbf{w_1}\in\mathbf{w}{:}\mathbf{w}\in\mathbf{v} \wedge \mathbf{w_1}\in\mathbf{u})\leftrightarrow\mathbf{w_1}\in\mathbf{u}$. Hence by lemma 1(ii), $\mathbf{w_1}{:}(\mathbf{w_1}\in\mathbf{w}{:}\mathbf{w}\in\mathbf{v} \wedge \mathbf{w_1}\in\mathbf{u}) \equiv \mathbf{w_1}{:}\mathbf{w_1}\in\mathbf{u}$, whence $\mathbf{x}{:}(\mathbf{x}\in\mathbf{w}{:}\mathbf{w}\in\mathbf{v} \wedge \mathbf{x}\in\mathbf{u}) \equiv \mathbf{w_1}{:}\mathbf{w_1}\in\mathbf{u}$ by lemmas 1(ii) and 4. Since $\mathbf{u}\neq V_1$ and $\mathbf{u}\neq V_2$, it follows by theorem 10(iii) that $\mathbf{w_1}{:}\mathbf{w_1}\in\mathbf{u} = \mathbf{w_1}{\cdot}\mathbf{w_1}\in\mathbf{u}$, whence $\text{E!}(\mathbf{w_1}{:}\mathbf{w_1}\in\mathbf{u})$ by the strength of identity. Hence $(\mathbf{x}{:}\mathbf{x}\in\mathbf{w}{:}\mathbf{w}\in\mathbf{v} \wedge \mathbf{x}\in\mathbf{u}) = \mathbf{w_1}{:}\mathbf{w_1}\in\mathbf{u}$, whence $(\mathbf{w}{:}\mathbf{w}\in\mathbf{v})\cap\mathbf{u} = \mathbf{w_1}{:}\mathbf{w_1}\in\mathbf{u}$ by the definition of $\cap$. By theorem 11(ii), $\mathbf{u}=\text{acc}(\mathbf{w_1}{:}\mathbf{w_1}\in\mathbf{u})$, whence $\mathbf{u}=\text{acc}((\mathbf{w}{:}\mathbf{w}\in\mathbf{v})\cap\mathbf{u})$.

7 (a) and (b) hold for arbitrary $\mathbf{u}\in\mathbf{w}{:}\mathbf{w}\in\mathbf{v}$. By lemma 4 we can generalize to get $\forall\mathbf{y}(\mathbf{y}\in\mathbf{w}{:}\mathbf{w}\in\mathbf{v} \rightarrow (\mathbf{y}=V_1 \vee \mathbf{y}=V_2 \rightarrow \mathbf{y}=\text{accum}((\mathbf{w}{:}\mathbf{w}\in\mathbf{v})\cap\mathbf{y})) \wedge (\mathbf{y}\neq V_1 \wedge \mathbf{y}\neq V_2 \rightarrow \mathbf{y}=\text{acc}((\mathbf{w}{:}\mathbf{w}\in\mathbf{v})\cap\mathbf{y})))$, which together with $S(\mathbf{w}{:}\mathbf{w}\in\mathbf{v})$ entails $H(\mathbf{w}{:}\mathbf{w}\in\mathbf{v})$ by the definition of H.

THEOREM 13. *Foundation for levels*

(i) *Let* $\exists\mathbf{u}\ \mathbf{u}\in\mathbf{x}$, *then* $\exists\mathbf{v}(\mathbf{v}\in\mathbf{x} \wedge \neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{v} \wedge \mathbf{w}\in\mathbf{x}))$.

(ii) *Let* $\exists\mathbf{u}\ A(\mathbf{u})$, *then* $\exists\mathbf{v}(A(\mathbf{v}) \wedge \neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{v} \wedge A(\mathbf{w})))$.

PROOF OF (i). By hypothesis $\mathbf{u}\in\mathbf{x}$ for some level $\mathbf{u}$. We tackle three cases separately: (i) $\neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{u} \wedge \mathbf{w}\in\mathbf{x})$, (ii) $\exists_1\mathbf{w}(\mathbf{w}\in\mathbf{u} \wedge \mathbf{w}\in\mathbf{x})$, and (iii) $m\mathbf{w}(\mathbf{w}\in\mathbf{u} \wedge \mathbf{w}\in\mathbf{x})$.

CASE (i) $\neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{u} \wedge \mathbf{w}\in\mathbf{x})$

It follows immediately that $\exists\mathbf{v}(\mathbf{v}\in\mathbf{x} \wedge \neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{v} \wedge \mathbf{w}\in\mathbf{x}))$.

CASE (ii) $\exists_1\mathbf{w}(\mathbf{w}\in\mathbf{u} \wedge \mathbf{w}\in\mathbf{x})$

By hypothesis $\mathbf{w_1}\in\mathbf{u} \wedge \mathbf{w_1}\in\mathbf{x}$ for some unique level $\mathbf{w_1}$. For a reductio suppose that $\mathbf{w_2}\in\mathbf{w_1}$ and $\mathbf{w_2}\in\mathbf{x}$ for some level $\mathbf{w_2}$. Since $\mathbf{w_2}\in\mathbf{w_1}$ and $\mathbf{w_1}\in\mathbf{u}$, it follows by theorem 8 that $\mathbf{w_2}\in\mathbf{u}$. Since $\mathbf{w_2}\in\mathbf{w_1}$, it follows by axiom 1(iii) that $\mathbf{w_2}\neq\mathbf{w_1}$. But $\mathbf{w_1}\in\mathbf{u}$, $\mathbf{w_1}\in\mathbf{x}$, $\mathbf{w_2}\in\mathbf{u}$, $\mathbf{w_2}\in\mathbf{x}$, and $\mathbf{w_2}\neq\mathbf{w_1}$ are together contrary to $\exists_1\mathbf{w}(\mathbf{w}\in\mathbf{u} \wedge \mathbf{w}\in\mathbf{x})$. Hence $\mathbf{w_1}\in\mathbf{x} \wedge \neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{w_1} \wedge \mathbf{w}\in\mathbf{x})$, whence $\exists\mathbf{v}(\mathbf{v}\in\mathbf{x} \wedge \neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{v} \wedge \mathbf{w}\in\mathbf{x}))$.

CASE (iii) $m\mathbf{w}(\mathbf{w}\in\mathbf{u} \wedge \mathbf{w}\in\mathbf{x})$

1 Since $m\mathbf{w}(\mathbf{w}\in\mathbf{u} \wedge \mathbf{w}\in\mathbf{x})$, it follows by lemmas 2(i) and 8(ii) that $\text{E!}(\mathbf{w}{\cdot}\mathbf{w}\in\mathbf{u} \wedge \mathbf{w}\in\mathbf{x})$. Hence by lemmas 4 and 8(ii), $\mathbf{w_1}\in\mathbf{w}{\cdot}(\mathbf{w}\in\mathbf{u} \wedge \mathbf{w}\in\mathbf{x}) \leftrightarrow (\mathbf{w_1}\in\mathbf{u} \wedge \mathbf{w_1}\in\mathbf{x})$, and also $M(\mathbf{w}{\cdot}\mathbf{w}\in\mathbf{u} \wedge \mathbf{w}\in\mathbf{x})$.

2 Since $m\mathbf{w}(\mathbf{w}\in\mathbf{u} \wedge \mathbf{w}\in\mathbf{x})$, it follows that $m\mathbf{w}\ \mathbf{w}\in\mathbf{u}$. Hence by lemmas 2(i) and 8(ii) $\mathsf{E}!(\mathbf{w}\cdot\mathbf{w}\in\mathbf{u})$. Hence by lemmas 4 and 8(ii), $\mathbf{w_1}\in\mathbf{w}\cdot\mathbf{w}\in\mathbf{u} \leftrightarrow \mathbf{w_1}\in\mathbf{u}$.

3 Since $M\mathbf{w}\cdot(\mathbf{w}\in\mathbf{u} \wedge \mathbf{w}\in\mathbf{x})$ and $(\mathbf{w_1}\in\mathbf{u} \wedge \mathbf{w_1}\in\mathbf{x})\rightarrow\mathbf{w_1}\in\mathbf{u}$, it follows that $\mathbf{w}\cdot(\mathbf{w}\in\mathbf{u} \wedge \mathbf{w}\in\mathbf{x}))\subseteq\mathbf{w}\cdot\mathbf{w}\in\mathbf{u}$ by the definition of $\subseteq$.

4 Since $m\mathbf{w}\ \mathbf{w}\in\mathbf{u}$, it follows by theorems 10(i) and 10(ii) that $\mathbf{u}\neq V_1$ and $\mathbf{u}\neq V_2$, whence by theorem 10(iii), $\mathbf{w}{:}\mathbf{w}\in\mathbf{u} = \mathbf{w}\cdot\mathbf{w}\in\mathbf{u}$. Hence by theorem 12, $H(\mathbf{w}\cdot\mathbf{w}\in\mathbf{u})$.

5 Since $\mathbf{w}\cdot(\mathbf{w}\in\mathbf{u} \wedge \mathbf{w}\in\mathbf{x})\subseteq\mathbf{w}\cdot\mathbf{w}\in\mathbf{u}$ and $H(\mathbf{w}\cdot\mathbf{w}\in\mathbf{u})$, it follows by lemmas 2(iv) and 8(ii), theorem 7 and the definition of $\cap$ that for some level $\mathbf{w_2}$, $\mathbf{w_2}\in\mathbf{w}\cdot(\mathbf{w}\in\mathbf{u} \wedge \mathbf{w}\in\mathbf{x}) \wedge \neg\exists\mathbf{z}(\mathbf{z}\in \mathbf{w}\cdot(\mathbf{w}\in\mathbf{u} \wedge \mathbf{w}\in\mathbf{x}) \wedge \mathbf{z}\in\mathbf{w_2})$, whence $\mathbf{w_2}\in\mathbf{u} \wedge \mathbf{w_2}\in\mathbf{x}$.

6 For a reductio suppose that $\mathbf{w_3}\in\mathbf{w_2}$ and $\mathbf{w_3}\in\mathbf{x}$ for some level $\mathbf{w_3}$. Since $\mathbf{w_3}\in\mathbf{w_2}$ and $\mathbf{w_2}\in\mathbf{u}$, it follows by theorem 8 that $\mathbf{w_3}\in\mathbf{u}$. Since $\mathbf{w_3}\in\mathbf{u}$ and $\mathbf{w_3}\in\mathbf{x}$, it follows that $\mathbf{w_3}\in\mathbf{w}\cdot(\mathbf{w}\in\mathbf{u} \wedge \mathbf{w}\in\mathbf{x})$. But $\mathbf{w_3}\in\mathbf{w}\cdot(\mathbf{w}\in\mathbf{u} \wedge \mathbf{w}\in\mathbf{x})$ and $\mathbf{w_3}\in\mathbf{w_2}$ are together contrary to $\neg\exists\mathbf{z}(\mathbf{z}\in\mathbf{w}\cdot(\mathbf{w}\in\mathbf{u} \wedge \mathbf{w}\in\mathbf{x}) \wedge \mathbf{z}\in\mathbf{w_2})$. Hence $\mathbf{w_2}\in\mathbf{x} \wedge \neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{w_2} \wedge \mathbf{w}\in\mathbf{x})$, whence $\exists\mathbf{v}(\mathbf{v}\in\mathbf{x} \wedge \neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{v} \wedge \mathbf{w}\in\mathbf{x}))$.

PROOF OF (ii). By hypothesis $A(\mathbf{u})$ for some level $\mathbf{u}$. We tackle two cases separately: (i) $\exists_1\mathbf{w}\ A(\mathbf{w})$, and (ii) $m\mathbf{w}\ A(\mathbf{w})$.

CASE (i) $\exists_1\mathbf{w}\ A(\mathbf{w})$

Suppose for a reductio that $\mathbf{w_1}\in\mathbf{u} \wedge A(\mathbf{w_1})$ for some level $\mathbf{w_1}$. Then by axiom 1(iii), $\mathbf{w_1}\neq\mathbf{u}$. But $A(\mathbf{u})$ and $A(\mathbf{w_1})$ and $\mathbf{w_1}\neq\mathbf{u}$ are together contrary to $\exists_1\mathbf{w}\ A(\mathbf{w})$. Hence $\neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{u} \wedge A(\mathbf{w}))$, whence $\exists\mathbf{v}(A(\mathbf{v}) \wedge \neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{v} \wedge A(\mathbf{w})))$.

CASE (ii) $m\mathbf{w}\ A(\mathbf{w})$

It follows by lemma 8(ii) that $m\mathbf{w}(E\mathbf{w} \wedge A(\mathbf{w}))$,whence $\mathsf{E}!(\mathbf{w_1}\cdot A(\mathbf{w_1}))$ by lemma 2(i). Hence $\mathbf{w_2}\in\mathbf{w_1}\cdot A(\mathbf{w_1})\leftrightarrow A(\mathbf{w_2})$ by lemmas 4 and 8(ii). Hence $\mathbf{u}\in\mathbf{w_1}\cdot A(\mathbf{w_1})$, whence by theorem 13(i), $\exists\mathbf{v}(\mathbf{v}\in\mathbf{w_1}\cdot A(\mathbf{w_1}) \wedge \neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{v} \wedge \mathbf{w}\in\mathbf{w_1}\cdot A(\mathbf{w_1})))$. Hence $\exists\mathbf{v}(A(\mathbf{v}) \wedge \neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{v} \wedge A(\mathbf{w})))$.

THEOREM 14. *Comparability of levels* $\mathbf{v}\in\mathbf{w} \vee \mathbf{v}=\mathbf{w} \vee \mathbf{w}\in\mathbf{v}$

PROOF.

1 For a reductio suppose for some $\mathbf{v}$, $\exists\mathbf{w}(\mathbf{v}\notin\mathbf{w} \wedge \mathbf{v}\neq\mathbf{w} \wedge \mathbf{w}\notin\mathbf{v})$. Then for some $\mathbf{v_1}$, $\exists\mathbf{w}(\mathbf{v_1}\notin\mathbf{w} \wedge \mathbf{v_1}\neq\mathbf{w} \wedge \mathbf{w}\notin\mathbf{v_1}) \wedge \neg\exists\mathbf{v_2}(\mathbf{v_2}\in\mathbf{v_1} \wedge \exists\mathbf{w}(\mathbf{v_2}\notin\mathbf{w} \wedge \mathbf{v_2}\neq\mathbf{w} \wedge \mathbf{w}\notin\mathbf{v_2}))$ by theorem 13(ii). Hence $\forall\mathbf{v_2}(\mathbf{v_2}\in\mathbf{v_1}\rightarrow\forall\mathbf{w}(\mathbf{v_2}\in\mathbf{w} \vee \mathbf{v_2}=\mathbf{w} \vee \mathbf{w}\in\mathbf{v_2}))$.

2 Since for some $\mathbf{w}$, $(\mathbf{v_1}\notin\mathbf{w} \wedge \mathbf{v_1}\neq\mathbf{w} \wedge \mathbf{w}\notin\mathbf{v_1})$, it follows by theorem 13(ii) that for some $\mathbf{w_1}$, $(\mathbf{v_1}\notin\mathbf{w_1} \wedge \mathbf{v_1}\neq\mathbf{w_1} \wedge \mathbf{w_1}\notin\mathbf{v_1}) \wedge \neg\exists\mathbf{w_2}(\mathbf{w_2}\in\mathbf{w_1} \wedge (\mathbf{v_1}\notin\mathbf{w_2} \wedge \mathbf{v_1}\neq\mathbf{w_2} \wedge \mathbf{w_2}\notin\mathbf{v_1}))$. Hence $\forall\mathbf{w_2}(\mathbf{w_2}\in\mathbf{w_1}\rightarrow(\mathbf{v_1}\in\mathbf{w_2} \vee \mathbf{v_1}=\mathbf{w_2} \vee \mathbf{w_2}\in\mathbf{v_1}))$. We shall prove $\forall\mathbf{w_3}(\mathbf{w_3}\in\mathbf{v_1}\leftrightarrow\mathbf{w_3}\in\mathbf{w_1})$.

3 For the $\rightarrow$ half, suppose $\mathbf{w_3}\in\mathbf{v_1}$. Since $\mathbf{w_1}\notin\mathbf{v_1}$, then $\mathbf{w_3}\neq\mathbf{w_1}$. For a reductio suppose $\mathbf{w_1}\in\mathbf{w_3}$. Then from $\mathbf{w_3}\in\mathbf{v_1}$ it follows that $\mathbf{w_1}\in\mathbf{v_1}$ by theorem 8. Contradiction. Hence $\mathbf{w_1}\notin\mathbf{w_3}$. Since $\forall\mathbf{v_2}(\mathbf{v_2}\in\mathbf{v_1}\rightarrow\forall\mathbf{w}(\mathbf{v_2}\in\mathbf{w} \vee \mathbf{v_2}=\mathbf{w} \vee \mathbf{w}\in\mathbf{v_2}))$ and $\mathbf{w_3}\in\mathbf{v_1}$ and $\mathbf{w_3}\neq\mathbf{w_1}$ and $\mathbf{w_1}\notin\mathbf{w_3}$, it follows that $\mathbf{w_3}\in\mathbf{w_1}$.

4 For the $\leftarrow$ half, suppose $\mathbf{w_3}\in\mathbf{w_1}$. Since $\mathbf{v_1}\notin\mathbf{w_1}$, then $\mathbf{w_3}\neq\mathbf{v_1}$. For a reductio suppose $\mathbf{v_1}\in\mathbf{w_3}$. Then from $\mathbf{w_3}\in\mathbf{w_1}$ it follows that $\mathbf{v_1}\in\mathbf{w_1}$ by theorem 8. Contradiction. Hence $\mathbf{v_1}\notin\mathbf{w_3}$. Since $\forall\mathbf{w_2}(\mathbf{w_2}\in\mathbf{w_1}\rightarrow(\mathbf{v_1}\in\mathbf{w_2} \vee \mathbf{v_1}=\mathbf{w_2} \vee \mathbf{w_2}\in\mathbf{v_1}))$ and $\mathbf{w_3}\in\mathbf{w_1}$ and $\mathbf{w_3}\neq\mathbf{v_1}$ and $\mathbf{v_1}\notin\mathbf{w_3}$, it follows that $\mathbf{w_3}\in\mathbf{v_1}$.

5 Since $\forall \mathbf{w_3}(\mathbf{w_3}{\in}\mathbf{v_1}{\leftrightarrow}\mathbf{w_3}{\in}\mathbf{w_1})$, it follows by lemma 1(ii) that $\mathbf{w}{:}\mathbf{w}{\in}\mathbf{v_1} \equiv \mathbf{w}{:}\mathbf{w}{\in}\mathbf{w_1}$. We shall prove (a) $\mathbf{v_1}{\neq}V_1 \wedge \mathbf{w_1}{\neq}V_1$, and (b) $\mathbf{v_1}{\neq}V_2 \wedge \mathbf{w_1}{\neq}V_2$.

6 For (a), for a reductio suppose $\mathbf{v_1}{=}V_1$. By theorem 10(i), $\mathbf{w}{:}\mathbf{w}{\in}\mathbf{v_1} \equiv O$, whence $\mathbf{w}{:}\mathbf{w}{\in}\mathbf{w_1} \equiv O$. By theorems 11(i) and 11(ii), $\mathbf{w_1}$=accum($\mathbf{w}{:}\mathbf{w}{\in}\mathbf{w_1}$) or $\mathbf{w_1}$=acc($\mathbf{w}{:}\mathbf{w}{\in}\mathbf{w_1}$), whence $\mathbf{w_1}$=accum(O) or $\mathbf{w_1}$=acc(O). For a subordinate reductio suppose $\mathbf{w_1}$=accum(O). Then $\mathbf{w_1}{=}\mathbf{v_1}$ by the definition of V_1. Contradiction. Hence $\mathbf{w_1}$=acc(O), whence $\mathbf{w_1}{=}\mathbf{v_1}$ by lemma 9(i). Contradiction. Hence $\mathbf{v_1}{\neq}V_1$. By similar reasoning, $\mathbf{w_1}{\neq}V_1$.

7 For (b), for a reductio suppose $\mathbf{v_1}{=}V_2$. Then by lemma 9(vii), $V_1{\in}\mathbf{v_1}$, and also $\mathbf{w}{:}\mathbf{w}{\in}\mathbf{v_1} = \imath\mathbf{w}(\mathbf{w}{\in}\mathbf{v_1}) = V_1$ by theorem 10(ii). Since $\forall \mathbf{w_3}(\mathbf{w_3}{\in}\mathbf{v_1}{\leftrightarrow}\mathbf{w_3}{\in}\mathbf{w_1})$, it follows that $V_1{\in}\mathbf{w_1}$. By theorems 11(i) and 11(ii), $\mathbf{w_1}$=accum($\mathbf{w}{:}\mathbf{w}{\in}\mathbf{w_1}$) or $\mathbf{w_1}$=acc($\mathbf{w}{:}\mathbf{w}{\in}\mathbf{w_1}$). Since $\mathbf{w}{:}\mathbf{w}{\in}\mathbf{v_1} \equiv \mathbf{w}{:}\mathbf{w}{\in}\mathbf{w_1}$, it follows that $\mathbf{w_1}$=accum(V_1) or $\mathbf{w_1}$=acc(V_1). For a subordinate reductio suppose $\mathbf{w_1}$=accum(V_1). Then $\mathbf{w_1}{=}\mathbf{v_1}$ by the definition of V_2. Contradiction. Hence $\mathbf{w_1}$=acc(V_1). Then $\mathbf{w_1}{=}V_1$ by lemma 9(i). But $V_1{\in}\mathbf{w_1}$ and $\mathbf{w_1}{=}V_1$ are together contrary to axiom 1(iii). Hence $\mathbf{v_1}{\neq}V_2$. By similar reasoning, $\mathbf{w_1}{\neq}V_2$.

8 We can now proceed to the reductio initiated in step 1. From (a) and (b) it follows that $\mathbf{v_1}$=acc($\mathbf{w}{:}\mathbf{w}{\in}\mathbf{v_1}$) and $\mathbf{w_1}$=acc($\mathbf{w}{:}\mathbf{w}{\in}\mathbf{w_1}$) by theorem 11(ii). Since $\mathbf{w}{:}\mathbf{w}{\in}\mathbf{v_1} \equiv \mathbf{w}{:}\mathbf{w}{\in}\mathbf{w_1}$, it follows that $\mathbf{v_1}$ = acc($\mathbf{w}{:}\mathbf{w}{\in}\mathbf{v_1}$) = acc($\mathbf{w}{:}\mathbf{w}{\in}\mathbf{w_1}$) = $\mathbf{w_1}$. Contradiction. Hence $\mathbf{v}{\in}\mathbf{w} \vee \mathbf{v}{=}\mathbf{w} \vee \mathbf{w}{\in}\mathbf{v}$.

THEOREM 15. *The lowest level principle*

(i) *Let* $\exists\mathbf{u}\ \mathbf{u}{\in}\mathbf{x}$, *then* $\exists_1\mathbf{v}(\mathbf{v}{\in}\mathbf{x} \wedge \neg\exists\mathbf{w}(\mathbf{w}{\in}\mathbf{v} \wedge \mathbf{w}{\in}\mathbf{x}))$.

(ii) *Let* $\exists\mathbf{u}\ A(\mathbf{u})$, *then* $\exists_1\mathbf{v}(A(\mathbf{v}) \wedge \neg\exists\mathbf{w}(\mathbf{w}{\in}\mathbf{v} \wedge A(\mathbf{w})))$.

PROOF OF (i). Since by hypothesis $\exists\mathbf{u}\ \mathbf{u}{\in}\mathbf{x}$, it follows by theorem 13(i) that $\mathbf{v}{\in}\mathbf{x} \wedge \neg\exists\mathbf{w}(\mathbf{w}{\in}\mathbf{v} \wedge \mathbf{w}{\in}\mathbf{x})$ for some level $\mathbf{v}$. For a reductio suppose that $\mathbf{v_1}{\in}\mathbf{x} \wedge \neg\exists\mathbf{w}(\mathbf{w}{\in}\mathbf{v_1} \wedge \mathbf{w}{\in}\mathbf{x})$ for some level $\mathbf{v_1}{\neq}\mathbf{v}$. Then by theorem 14 it follows that $\mathbf{v}{\in}\mathbf{v_1} \vee \mathbf{v_1}{\in}\mathbf{v}$. But if $\mathbf{v}{\in}\mathbf{v_1}$ then $\mathbf{v}{\in}\mathbf{v_1} \wedge \mathbf{v}{\in}\mathbf{x}$, contrary to $\neg\exists\mathbf{w}(\mathbf{w}{\in}\mathbf{v_1} \wedge \mathbf{w}{\in}\mathbf{x}))$. Similarly, if $\mathbf{v_1}{\in}\mathbf{v}$ then $\mathbf{v_1}{\in}\mathbf{v} \wedge \mathbf{v_1}{\in}\mathbf{x}$, contrary to $\neg\exists\mathbf{w}(\mathbf{w}{\in}\mathbf{v} \wedge \mathbf{w}{\in}\mathbf{x})$. Contradiction. Hence $\exists_1\mathbf{v}(\mathbf{v}{\in}\mathbf{x} \wedge \neg\exists\mathbf{w}(\mathbf{w}{\in}\mathbf{v} \wedge \mathbf{w}{\in}\mathbf{x}))$.

PROOF OF (ii). The proof runs parallel to the proof of theorem 15(i), but uses theorem 13(ii) instead of theorem 13(i).

THEOREM 16. *Uniqueness of histories*

(i) *Let* V_1=accum($\mathbf{h}$), *then* $\mathbf{h} \equiv \mathbf{w}{:}\mathbf{w}{\in}V_1$.

(ii) *Let* V_2=accum($\mathbf{h}$), *then* $\mathbf{h} = \mathbf{w}{:}\mathbf{w}{\in}V_2$.

(iii) *Let* $\mathbf{v}{\neq}V_1$ and $\mathbf{v}{\neq}V_2$ and $\mathbf{v}$=acc($\mathbf{h}$), *then* $\mathbf{h} = \mathbf{w}{:}\mathbf{w}{\in}\mathbf{v}$.

PROOF OF (i).

1 It follows from the hypothesis that $\mathsf{E}!V_1$ by the strength of identity, whence $\mathbf{z}{\in}V_1{\to}U\mathbf{z}$ by lemma 9(ii).

2 For a reductio suppose $\mathbf{h}{\not\equiv}\mathbf{w}{:}\mathbf{w}{\in}V_1$. Then $\mathbf{h}{\not\equiv}O$ by theorem 10(i), whence $S\mathbf{h}$ by the definition of H, and $M\mathbf{h} \wedge E\mathbf{h}$ by the definition of S. Hence $\mathbf{h}{\sqsubseteq}\mathbf{h}$ by lemma 6(ii). By the definition of accum, V_1 = accum($\mathbf{h}$) = $\mathbf{z}{\cdot}(U\mathbf{z} \vee \mathbf{z}{\in}\mathbf{h} \vee \mathbf{z}{\sqsubseteq}\mathbf{h})$, whence $\mathbf{h}{\in}V_1$ by lemma 4. Hence $U\mathbf{h}$, whence $\neg M\mathbf{h}$ by the definition of U. Contradiction. Hence $\mathbf{h} \equiv \mathbf{w}{:}\mathbf{w}{\in}V_1$.

PROOF OF (ii).

1 It follows from the hypothesis that $\mathsf{E}!V_2$ by the strength of identity. By theorem 10(ii), $\mathbf{w}{:}\mathbf{w}\in V_2 = V_1$, whence $\mathsf{E}!V_1$ by the strength of identity. Hence MV_1 by lemma 9(ii).

2 For a reductio suppose $\mathbf{h}\equiv O$. Then by the definition of V_1, $V_2 = \text{accum}(\mathbf{h}) = \text{accum}(O) = V_1$, contrary to lemma 9(viii). Hence $\mathbf{h}\not\equiv O$, whence $S\mathbf{h}$ by the definition of H, and $M\mathbf{h} \wedge E\mathbf{h}$ by the definition of S. Hence $\mathbf{h}\subseteq\mathbf{h}$ by lemma 6(ii). By the definition of accum, $V_2 = \text{accum}(\mathbf{h}) = \mathbf{z}\cdot(U\mathbf{z} \vee \mathbf{z}\in\mathbf{h} \vee \mathbf{z}\subseteq\mathbf{h})$, whence $\mathbf{h}\in V_2$ by lemma 4. Hence by lemma 9(v), $U\mathbf{h} \vee \mathbf{h}\subseteq V_1$. Since $M\mathbf{h}$, it follows that $\neg U\mathbf{h}$ by the definition of U, whence $\mathbf{h}\subseteq V_1$. Hence $\mathbf{z}\in\mathbf{h}\rightarrow U\mathbf{z}$ by the definition of $\subseteq$ and lemma 9(ii).

3 For a reductio suppose $\mathbf{h}\neq\mathbf{w}{:}\mathbf{w}\in V_2$. Then $\mathbf{h}\neq V_1$, whence $\mathbf{h}\subset V_1$ by the definition of $\subset$. Hence for some $\mathbf{z_1}$, $\mathbf{z_1}\notin\mathbf{h} \wedge \mathbf{z_1}\in V_1$ by lemma 6(iii). By lemma 9(vii) $V_1\in V_2$, whence $UV_1 \vee V_1\in\mathbf{h} \vee V_1\subseteq\mathbf{h}$ by lemma 4. Since MV_1, it follows that $\neg UV_1$ by the definition of U. Hence $V_1\in\mathbf{h} \vee V_1\subseteq\mathbf{h}$. For a subordinate reductio suppose $V_1\in\mathbf{h}$. Then UV_1, whence $\neg MV_1$ by the definition of U. Contradiction. Hence $V_1\subseteq\mathbf{h}$, whence $\mathbf{z}\in V_1\rightarrow\mathbf{z}\in\mathbf{h}$ by the definition of $\subseteq$. Contradiction. Hence $\mathbf{h} = \mathbf{w}{:}\mathbf{w}\in V_2$.

PROOF OF (iii).

1 It follows from the hypothesis and theorem 3(ii) that $M\mathbf{h} \wedge \mathbf{h}\neq V_1$, whence $\mathsf{E}!\mathbf{h}$ by lemma 3(i).

2 For a reductio suppose $\mathbf{h}\neq\mathbf{w}\cdot\mathbf{w}\in\mathbf{v}$. Then by the hypothesis and theorem 15(ii), there is a unique level $\mathbf{v_1}$ such that for some history $\mathbf{h_1}$, $\mathbf{v_1}\neq V_1 \wedge \mathbf{v_1}\neq V_2 \wedge \mathbf{v_1}=\text{acc}(\mathbf{h_1}) \wedge \mathbf{h_1}\neq\mathbf{w}\cdot\mathbf{w}\in\mathbf{v_1}$, and $\neg\exists\mathbf{w_1}(\mathbf{w_1}\in\mathbf{v_1} \wedge \exists\mathbf{x}(H\mathbf{x} \wedge \mathbf{w_1}\neq V_1 \wedge \mathbf{w_1}\neq V_2 \wedge \mathbf{w_1}=\text{acc}(\mathbf{x}) \wedge \mathbf{x}\neq\mathbf{w}\cdot\mathbf{w}\in\mathbf{w_1})$. Hence $M\mathbf{h_1} \wedge \mathbf{h_1}\neq V_1$ by theorem 3(ii). By the definition of acc, $\mathbf{v_1} = \text{acc}(\mathbf{h_1}) = \mathbf{z}\cdot(U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{h_1} \wedge (\mathbf{z}\in\mathbf{y} \vee \mathbf{z}\subseteq\mathbf{y})))$, whence by lemma 4, $\mathbf{z}\in\mathbf{v_1}\leftrightarrow(E\mathbf{z} \wedge (U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{h_1} \wedge (\mathbf{z}\in\mathbf{y} \vee \mathbf{z}\subseteq\mathbf{y}))))$. We shall prove that $\mathbf{w_2}\in\mathbf{h_1}\leftrightarrow\mathbf{w_2}\in\mathbf{v_1}$.

3 The $\rightarrow$ half is immediate by theorem 3(i).

4 For the $\leftarrow$ half, suppose $\mathbf{w_2}\in\mathbf{v_1}$. Then $\mathsf{E}!\mathbf{w_2}$ by axiom 1(i). For a reductio suppose $\mathbf{w_3}\in\mathbf{h_1}\rightarrow\mathbf{w_3}\in\mathbf{w_2}$. We tackle three cases separately, deriving a contradiction for each: (i) $\mathbf{w_2}=V_1$, (ii) $\mathbf{w_2}=V_2$, and (iii) $\mathbf{w_2}\neq V_1$ and $\mathbf{w_2}\neq V_2$.

5 For case (i), since $M\mathbf{h_1}$, it follows by the definition of M that $\mathbf{x}\in\mathbf{h_1}$ for some $\mathbf{x}$. Since $\mathbf{h_1}\neq V_1$, it follows by theorem 6 that $V\mathbf{x}$. By supposition $\mathbf{w_3}\in\mathbf{h_1}\rightarrow\mathbf{w_3}\in\mathbf{w_2}$, whence $\mathbf{x}\in V_1$. But by theorem 10(i) $\neg\exists\mathbf{w}\ \mathbf{w}\in V_1$. Contradiction.

6 For case (ii), since $M\mathbf{h_1}$, it follows by the definition of M that $\mathbf{x}\in\mathbf{h_1}$ and $\mathbf{y}\in\mathbf{h_1}$, for some $\mathbf{x}$, $\mathbf{y}$ where $\mathbf{x}\neq\mathbf{y}$. Since $\mathbf{h_1}\neq V_1$, it follows by theorem 6 that $V\mathbf{x}$ and $V\mathbf{y}$. By supposition $\mathbf{w_3}\in\mathbf{h_1}\rightarrow\mathbf{w_3}\in\mathbf{w_2}$, and so $\mathbf{x}\in V_2$ and $\mathbf{y}\in V_2$. But by theorem 10(ii), $\exists_1\mathbf{w}\ \mathbf{w}\in V_2$. Contradiction.

7 For case (iii), by theorem 10(iii) it follows that $\mathbf{w_4}{:}\mathbf{w_4}\in\mathbf{w_2} = \mathbf{w_4}\cdot\mathbf{w_4}\in\mathbf{w_2}$, whence $\mathbf{y}\in\mathbf{w_4}\cdot\mathbf{w_4}\in\mathbf{w_2}\leftrightarrow(V\mathbf{y} \wedge \mathbf{y}\in\mathbf{w_2})$ by lemmas 4 and 8(ii). By theorem 11(ii), $\mathbf{w_2} = \text{acc}(\mathbf{w_4}\cdot\mathbf{w_4}\in\mathbf{w_2})$, whence $\mathbf{w_2} = \mathbf{z}\cdot(U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{w_4}\cdot\mathbf{w_4}\in\mathbf{w_2} \wedge (\mathbf{z}\in\mathbf{y} \vee \mathbf{z}\subseteq\mathbf{y})))$ by the definition of acc. Hence $\mathbf{z}\in\mathbf{w_2}\leftrightarrow(E\mathbf{z} \wedge (U\mathbf{z} \vee \exists\mathbf{w}(\mathbf{w}\in\mathbf{w_2} \wedge (\mathbf{z}\in\mathbf{w} \vee \mathbf{z}\subseteq\mathbf{w}))))$ by lemma 4. Since $\mathbf{h_1}\neq V_1$, it follows by theorem 6 that $\mathbf{y}\in\mathbf{h_1}\rightarrow V\mathbf{y}$. Since (from step 2) $\mathbf{z}\in\mathbf{v_1}\rightarrow(E\mathbf{z} \wedge (U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{h_1} \wedge (\mathbf{z}\in\mathbf{y} \vee \mathbf{z}\subseteq\mathbf{y}))))$ and (by supposition) $\mathbf{w_3}\in\mathbf{h_1}\rightarrow\mathbf{w_3}\in\mathbf{w_2}$, it follows that $\mathbf{z}\in\mathbf{v_1}\rightarrow(E\mathbf{z} \wedge (U\mathbf{z} \vee \exists\mathbf{w}(\mathbf{w}\in\mathbf{w_2} \wedge (\mathbf{z}\in\mathbf{w} \vee \mathbf{z}\subseteq\mathbf{w}))))$, whence $\mathbf{z}\in\mathbf{v_1}\rightarrow\mathbf{z}\in\mathbf{w_2}$. But $\mathbf{w_2}\in\mathbf{v_1}$, so $\mathbf{w_2}\in\mathbf{w_2}$, contrary to axiom 1(iii).

8 Since each case leads to a contradiction, it follows that $\mathbf{w}_5\in\mathbf{h}_1 \wedge \mathbf{w}_5\notin\mathbf{w}_2$ for some $\mathbf{w}_5$, whence by theorem 14 $\mathbf{w}_2=\mathbf{w}_5 \vee \mathbf{w}_2\in\mathbf{w}_5$. Suppose $\mathbf{w}_2=\mathbf{w}_5$. Since $\mathbf{w}_5\in\mathbf{h}_1$, it follows that $\mathbf{w}_2\in\mathbf{h}_1$.

9 Taking the other alternative, suppose $\mathbf{w}_2\in\mathbf{w}_5$. We tackle three cases separately: (a) $\mathbf{w}_5=V_1$, (b) $\mathbf{w}_5=V_2$ and (c) $\mathbf{w}_5\neq V_1$ and $\mathbf{w}_5\neq V_2$, showing in each case that $\mathbf{w}_2\in\mathbf{h}_1$.

10 For case (a), by theorem 10(i) $\neg\exists\mathbf{w}\ \mathbf{w}\in V_1$, whence $\mathbf{w}_2\notin\mathbf{w}_5$. But also $\mathbf{w}_2\in\mathbf{w}_5$. By the tautology $A\wedge\neg A\rightarrow B$ it follows that $\mathbf{w}_2\in\mathbf{h}_1$.

11 For case (b), since $\mathbf{w}_2\in\mathbf{w}_5$, it follows that $\mathbf{w}_2\in V_2$. Hence by theorem 10(ii), $\mathbf{w}_2=V_1$. Since $\mathbf{w}_5\in\mathbf{h}_1$, it follows that $V_2\in\mathbf{h}_1$. Since $\mathbf{h}_1\neq V_1$, it follows by the reasoning in steps 1–5 of case (ii) of theorem 6 that $V_1\in\mathbf{h}_1$, whence $\mathbf{w}_2\in\mathbf{h}_1$.

12 For case (c), since $\mathbf{h}_1\neq V_1$ and $\mathbf{w}_5\in\mathbf{h}_1$ and $\mathbf{w}_5\neq V_1$ and $\mathbf{w}_5\neq V_2$, it follows by theorems 5(i), 5(ii) and 5(iii), and 6 that $\mathbf{w}_5=\mathrm{acc}(\mathbf{h}_1\cap\mathbf{w}_5)$ and $\mathsf{E}!(\mathbf{h}_1\cap\mathbf{w}_5)$ and $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\in\mathbf{h}_1 \wedge \mathbf{z}\in\mathbf{w}_5)$ and $H(\mathbf{h}_1\cap\mathbf{w}_5)$. Since $\mathbf{w}_5\in\mathbf{h}_1$ and $\mathbf{v}_1=\mathrm{acc}(\mathbf{h}_1)$, it follows by theorem 3(i) that $\mathbf{w}_5\in\mathbf{v}_1$. Hence $\neg\exists\mathbf{x}(H\mathbf{x} \wedge \mathbf{w}_5\neq V_1 \wedge \mathbf{w}_5\neq V_2 \wedge \mathbf{w}_5=\mathrm{acc}(\mathbf{x}) \wedge \mathbf{x}\neq\mathbf{w}\cdot\mathbf{w}\in\mathbf{w}_5)$ by step 2. Since $\mathsf{E}!(\mathbf{h}_1\cap\mathbf{w}_5)$ and $H(\mathbf{h}_1\cap\mathbf{w}_5)$ and $\mathbf{w}_5\neq V_1$ and $\mathbf{w}_5\neq V_2$ and $\mathbf{w}_5=\mathrm{acc}(\mathbf{h}_1\cap\mathbf{w}_5)$, it follows that $\mathbf{h}_1\cap\mathbf{w}_5 = \mathbf{w}\cdot\mathbf{w}\in\mathbf{w}_5$. Since $\mathbf{w}_2\in\mathbf{w}_5$, it follows by lemmas 4 and 8(ii) that $\mathbf{w}_2\in(\mathbf{h}_1\cap\mathbf{w}_5)$. Since $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\in\mathbf{h}_1 \wedge \mathbf{z}\in\mathbf{w}_5)$ it follows by the definition of $\cap$ and lemma 2(ii) that $\mathbf{h}_1\cap\mathbf{w}_5 = \mathbf{z}\cdot(\mathbf{z}\in\mathbf{h}_1 \wedge \mathbf{z}\in\mathbf{w}_5)$. Hence $\mathbf{w}_2\in\mathbf{z}\cdot(\mathbf{z}\in\mathbf{h}_1 \wedge \mathbf{z}\in\mathbf{w}_5)$, whence by lemma 4 $\mathbf{w}_2\in\mathbf{h}_1$.

13 We can now proceed to the reductio initiated in step 2. Since $\mathsf{E}!\mathbf{v}_1$ and $\mathbf{v}_1\neq V_1$ and $\mathbf{v}_1\neq V_2$, it follows by theorem 10(iii) that $\mathbf{w}{:}\mathbf{w}\in\mathbf{v}_1 = \mathbf{w}\cdot\mathbf{w}\in\mathbf{v}_1$. Hence $M(\mathbf{w}\cdot\mathbf{w}\in\mathbf{v}_1)$. by lemma 4. Since $\mathbf{w}_2\in\mathbf{h}_1\leftrightarrow\mathbf{w}_2\in\mathbf{v}_1$, it follows that $\mathbf{w}_2\in\mathbf{h}_1\leftrightarrow\mathbf{w}_2\in\mathbf{w}\cdot\mathbf{w}\in\mathbf{v}_1$ by lemmas 4 and 8(ii). Since $\mathbf{h}_1\neq V_1$, it follows by theorem 6 that $\mathbf{x}\in\mathbf{h}_1\rightarrow V\mathbf{x}$. Also $\mathbf{x}\in\mathbf{w}\cdot\mathbf{w}\in\mathbf{v}_1\rightarrow V\mathbf{x}$ by lemma 4. Hence $\mathbf{x}\in\mathbf{h}_1\leftrightarrow\mathbf{x}\in\mathbf{w}\cdot\mathbf{w}\in\mathbf{v}_1$. Since $M\mathbf{h}_1$ and $M(\mathbf{w}\cdot\mathbf{w}\in\mathbf{v}_1)$ and $\mathbf{x}\in\mathbf{h}_1\leftrightarrow\mathbf{x}\in\mathbf{w}\cdot\mathbf{w}\in\mathbf{v}_1$, it follows by axiom 1(ii) that $\mathbf{h}_1 = \mathbf{w}\cdot\mathbf{w}\in\mathbf{v}_1$. Contradiction. Hence $\mathbf{h} = \mathbf{w}\cdot\mathbf{w}\in\mathbf{v}$, whence $\mathsf{E}!(\mathbf{w}\cdot\mathbf{w}\in\mathbf{v})$. It follows by lemmas 2(i) and 2(ii) that $\mathbf{h} = \mathbf{w}{:}\mathbf{w}\in\mathbf{v}$.

THEOREM 17. *Sets and levels I*

(i) $S\mathbf{x}\leftrightarrow\mathsf{E}!V^*(\mathbf{x})$

(ii) $\mathbf{x}\notin V^*(\mathbf{x})$

(iii) $V^*(\mathbf{u})=\mathbf{u}$

(iv) Let $S\mathbf{x} \wedge S\mathbf{y} \wedge \mathbf{x}\in\mathbf{y}$, then $V^*(\mathbf{x})\in V^*(\mathbf{y})$.

(v) Let $S\mathbf{x} \wedge S\mathbf{y} \wedge \mathbf{x}\subseteq\mathbf{y}$, then $V^*(\mathbf{x})\subseteq V^*(\mathbf{y})$.

(vi) $S\mathbf{x}\leftrightarrow(M\mathbf{x} \wedge \exists\mathbf{u}\forall\mathbf{y}((S\mathbf{y} \wedge \mathbf{y}\in\mathbf{x})\rightarrow V^*(\mathbf{y})\in\mathbf{u}))$

PROOF OF (i).

1 For the $\rightarrow$ half, suppose $S\mathbf{x}$. Then by axiom 2(iv) $\exists\mathbf{u}\ \mathbf{x}\subseteq\mathbf{u}$. Hence by theorem 15(ii), $\exists_1\mathbf{v}(\mathbf{x}\subseteq\mathbf{v} \wedge \neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{v} \wedge \mathbf{x}\subseteq\mathbf{w}))$, whence $\mathsf{E}!\imath\mathbf{v}(\mathbf{x}\subseteq\mathbf{v} \wedge \neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{v} \wedge \mathbf{x}\subseteq\mathbf{w}))$. Hence $\mathsf{E}!V^*(\mathbf{x})$ by the definition of $V^*(\mathbf{x})$.

2 For the $\leftarrow$ half, suppose $\mathsf{E}!V^*(\mathbf{x})$. Then by the definition of $V^*(\mathbf{x})$ it follows that $V(V^*(\mathbf{x})) \wedge \mathbf{x}\subseteq V^*(\mathbf{x})$, whence $S\mathbf{x}$ by axiom 2(iii) and lemma 8(iii).

PROOF OF (ii).

1 Suppose $\neg S\mathbf{x}$. Then $\neg\mathsf{E}!V^*(\mathbf{x})$ by theorem 17(i), whence $\mathbf{x}\notin V^*(\mathbf{x})$ by axiom 1(i).

2 Suppose instead $S\mathbf{x}$. Then $\mathsf{E}!V^*(\mathbf{x})$ by theorem 17(i), whence $V(V^*(\mathbf{x}))$ by the definition of $V^*(\mathbf{x})$. For a reductio suppose $\mathbf{x}\in V^*(\mathbf{x})$. We tackle three cases separately—(i) $V^*(\mathbf{x})=V_1$, (ii) $V^*(\mathbf{x})=V_2$ and (iii) $V^*(\mathbf{x})\neq V_1$ and $V^*(\mathbf{x})\neq V_2$—deriving a contradiction in each case.

3 In case (i), $V^*(\mathbf{x})=V_1$. Then $U\mathbf{x}$ by lemma 9(ii). Hence by the definitions of U and S, $\neg S\mathbf{x}$. Contradiction.

4 In case (ii), $V^*(\mathbf{x})=V_2$. Hence $\mathsf{E}!V_1$ by lemma 9(vi). Since $\mathbf{x}\in V^*(\mathbf{x})$, it follows by lemma 9(v) that $U\mathbf{x} \vee \mathbf{x}\subseteq V_1$. Since $S\mathbf{x}$, it follows by the definitions of U and S that $\mathbf{x}\subseteq V_1$. By lemma 9(vii) and the definition of V, $V_1\in V^*(\mathbf{x}) \wedge V(V_1)$. But by the definition of $V^*(\mathbf{x})$, $\neg\exists\mathbf{u}(\mathbf{u}\in V^*(\mathbf{x}) \wedge \mathbf{x}\subseteq\mathbf{u})$. Contradiction.

5 In case (iii), $V^*(\mathbf{x})\neq V_1$ and $V^*(\mathbf{x})\neq V_2$. It follows that $\mathbf{w}{:}\mathbf{w}\in V^*(\mathbf{x}) = \mathbf{w}\cdot\mathbf{w}\in V^*(\mathbf{x})$ by theorem 10(iii). Since $\mathbf{x}\in V^*(\mathbf{x})$, it follows that $\mathbf{x}\in\mathrm{acc}(\mathbf{w}\cdot\mathbf{w}\in V^*(\mathbf{x}))$ by theorem 11(ii). By the definition of acc, $\mathrm{acc}(\mathbf{w}\cdot\mathbf{w}\in V^*(\mathbf{x})) = \mathbf{z}\cdot(U\mathbf{z} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{w}\cdot\mathbf{w}\in V^*(\mathbf{x}) \wedge (\mathbf{z}\in\mathbf{y} \vee \mathbf{z}\subseteq\mathbf{y})))$, whence $U\mathbf{x} \vee \exists\mathbf{y}(\mathbf{y}\in\mathbf{w}\cdot\mathbf{w}\in V^*(\mathbf{x}) \wedge (\mathbf{x}\in\mathbf{y} \vee \mathbf{x}\subseteq\mathbf{y}))$ by lemma 4. Since $S\mathbf{x}$, it follows that $\neg U\mathbf{x}$ by the definitions of U and S. Hence by lemma 4, $\mathbf{x}\in\mathbf{v} \vee \mathbf{x}\subseteq\mathbf{v}$ for some level $\mathbf{v}\in V^*(\mathbf{x})$, whence $\mathbf{x}\subseteq\mathbf{v}$ by the corollary of theorem 8 and the definition of S. But by the definition of $V^*(\mathbf{x})$, $\neg\exists\mathbf{u}(\mathbf{u}\in V^*(\mathbf{x}) \wedge \mathbf{x}\subseteq\mathbf{u})$. Contradiction.

6 Since a contradiction is derivable in each case, it follows that $\mathbf{x}\notin V^*(\mathbf{x})$.

PROOF OF (iii). By lemma 8(iii), $S\mathbf{u}$, whence $\mathsf{E}!V^*(\mathbf{u}) \wedge V(V^*(\mathbf{u}))$ by theorem 17(i) and the definition of $V^*(\mathbf{u})$. By theorem 17(ii), $\mathbf{u}\notin V^*(\mathbf{u})$. For a reductio suppose $V^*(\mathbf{u})\in\mathbf{u}$. By the definition of $V^*(\mathbf{u})$, $\mathbf{u}\subseteq V^*(\mathbf{u})$, whence by theorem 9, $\mathbf{u}\in\mathbf{u}$, contrary to axiom 1(iii). Hence $V^*(\mathbf{u})\notin\mathbf{u}$, whence $V^*(\mathbf{u})=\mathbf{u}$ by theorem 14.

PROOF OF (iv). By the hypothesis and theorem 17(i), $\mathsf{E}!V^*(\mathbf{x}) \wedge \mathsf{E}!V^*(\mathbf{y})$, whence by the definitions of $V^*(\mathbf{x})$ and $V^*(\mathbf{y})$, $V(V^*(\mathbf{x})) \wedge V(V^*(\mathbf{y}))$. Also by the definition of $V^*(\mathbf{y})$, $\mathbf{y}\subseteq V^*(\mathbf{y})$, whence by the hypothesis and the definition of $\subseteq$, $\mathbf{x}\in V^*(\mathbf{y})$. For a reductio suppose $V^*(\mathbf{x})=V^*(\mathbf{y})$. Hence $\mathbf{x}\in V^*(\mathbf{x})$, contrary to theorem 17(ii). Hence $V^*(\mathbf{x})\neq V^*(\mathbf{y})$. For a reductio suppose $V^*(\mathbf{y})\in V^*(\mathbf{x})$. Then by theorem 8 $\mathbf{x}\in V^*(\mathbf{x})$, contrary to theorem 17(ii). Hence $V^*(\mathbf{y})\notin V^*(\mathbf{x})$, whence by theorem 14 $V^*(\mathbf{x})\in V^*(\mathbf{y})$.

PROOF OF (v). By the hypothesis and theorem 17(i), $\mathsf{E}!V^*(\mathbf{x}) \wedge \mathsf{E}!V^*(\mathbf{y})$, whence by the definitions of $V^*(\mathbf{x})$ and $V^*(\mathbf{y})$, $V(V^*(\mathbf{x})) \wedge V(V^*(\mathbf{y}))$. Also by the definition of $V^*(\mathbf{y})$, $\mathbf{y}\subseteq V^*(\mathbf{y})$, whence by the hypothesis and the definition of $\subseteq$, $\mathbf{x}\subseteq V^*(\mathbf{y})$. For a reductio suppose $V^*(\mathbf{y})\in V^*(\mathbf{x})$. By the definition of $V^*(\mathbf{x})$, $\neg\exists\mathbf{w}(\mathbf{w}\in V^*(\mathbf{x}) \wedge \mathbf{x}\subseteq\mathbf{w})$. Contradiction. Hence $V^*(\mathbf{y})\notin V^*(\mathbf{x})$. Hence by theorem 14 $V^*(\mathbf{x})\in V^*(\mathbf{y}) \vee V^*(\mathbf{x})=V^*(\mathbf{y})$. Suppose $V^*(\mathbf{x})\in V^*(\mathbf{y})$. Then $V^*(\mathbf{x})\subseteq V^*(\mathbf{y})$ by lemma 8(i) and the corollary of theorem 8. Suppose $V^*(\mathbf{x})=V^*(\mathbf{y})$. Then $V^*(\mathbf{x})\subseteq V^*(\mathbf{y})$ by lemmas 6(ii) and 8(i).

PROOF OF (vi).

1 For the $\rightarrow$ half, suppose $S\mathbf{x}$. Then $M\mathbf{x} \wedge \mathsf{E}!\mathbf{x}$ by the definition of S and lemma 3(i), whence $\exists\mathbf{x}V\mathbf{x}$ by lemma 9(iii), the strength of identity, and the definition of V. Suppose $\neg\exists\mathbf{z}(S\mathbf{z} \wedge \mathbf{z}\in\mathbf{x})$. Then $\exists\mathbf{u}\forall\mathbf{y}((S\mathbf{y} \wedge \mathbf{y}\in\mathbf{x})\rightarrow V^*(\mathbf{y})\in\mathbf{u})$.

2 Suppose instead $\exists\mathbf{z}(S\mathbf{z} \wedge \mathbf{z}{\in}\mathbf{x})$. Consider an arbitrary $\mathbf{z_1}$ such that $S\mathbf{z_1} \wedge \mathbf{z_1}{\in}\mathbf{x}$. By axiom 2(iv) $\mathbf{x}{\subseteq}\mathbf{v}$ for some level $\mathbf{v}$. Then $\mathbf{z_1}{\in}\mathbf{v}$ by the definition of $\subseteq$, whence by theorems 17(iii) and 17(iv), and lemma 8(iii), $V^*(\mathbf{z}_1){\in}\mathbf{v}$. Since $\mathbf{z}_1$ is arbitrary, we can generalize to get $\forall\mathbf{y}((S\mathbf{y} \wedge \mathbf{y}{\in}\mathbf{x}){\to}V^*(\mathbf{y}){\in}\mathbf{v})$, whence $\exists\mathbf{u}\forall\mathbf{y}((S\mathbf{y} \wedge \mathbf{y}{\in}\mathbf{x}){\to}V^*(\mathbf{y}){\in}\mathbf{u})$.

3 For the $\leftarrow$ half, suppose $M\mathbf{x}$ and $\forall\mathbf{y}((S\mathbf{y} \wedge \mathbf{y}{\in}\mathbf{x}){\to}V^*(\mathbf{y}){\in}\mathbf{v})$ for some level $\mathbf{v}$. Then $\mathsf{E}!\mathbf{x}$ by lemma 3(i), whence $\mathsf{E}!V_1$ by lemma 9(iii) and the strength of identity. Suppose $\neg\exists\mathbf{z}(S\mathbf{z} \wedge \mathbf{z}{\in}\mathbf{x})$. Then $\forall\mathbf{z}(\mathbf{z}{\in}\mathbf{x}{\to}U\mathbf{z})$ by lemmas 3(iii) and 5(i), whence $\forall\mathbf{z}(\mathbf{z}{\in}\mathbf{x}{\to}\mathbf{z}{\in}V_1)$ by lemma 9(ii). Hence $\mathbf{x}{\subseteq}V_1$ by the definition of $\subseteq$, whence $S\mathbf{x}$ by the definition of V, and theorem 1(ii).

4 Suppose instead $\exists\mathbf{z}(S\mathbf{z} \wedge \mathbf{z}{\in}\mathbf{x})$. Consider an arbitrary $\mathbf{z_1}$ such that $S\mathbf{z_1} \wedge \mathbf{z_1}{\in}\mathbf{x}$. Then $V^*(\mathbf{z_1}){\in}\mathbf{v}$, whence $\mathsf{E}!V^*(\mathbf{z_1})$ by axiom 1(i). Since by the definition of $V^*(\mathbf{z_1})$, $\mathbf{z_1}{\subseteq}V^*(\mathbf{z_1})$, it follows that $\mathbf{z_1}{\in}\mathbf{v}$ by theorem 9. Since $\mathbf{z_1}$ is arbitrary, we can generalize to get $\forall\mathbf{z}((S\mathbf{z} \wedge \mathbf{z}{\in}\mathbf{x}){\to}\mathbf{z}{\in}\mathbf{v})$, whence $\forall\mathbf{z}(\mathbf{z}{\in}\mathbf{x}{\to}\mathbf{z}{\in}\mathbf{v})$ by lemmas 3(iii), 5(i) and 8(iv). Hence $\mathbf{x}{\subseteq}\mathbf{v}$ by the definition of $\subseteq$, whence $S\mathbf{x}$ by theorem 1(ii).

THEOREM 18. *Foundation for multitudes* *Let* $M\mathbf{x}$, *then* $\exists\mathbf{y}(\mathbf{y}{\in}\mathbf{x} \wedge \mathbf{x}{\cap}\mathbf{y}{\equiv}O)$.

PROOF.

1 Suppose $U\mathbf{y} \wedge \mathbf{y}{\in}\mathbf{x}$ for some $\mathbf{y}$. Then $\neg M\mathbf{y}$ by the definition of U. So by axiom 1(iv) $\neg\exists\mathbf{z}\ \mathbf{z}{\in}\mathbf{y}$. Hence $\neg\exists\mathbf{z}(\mathbf{z}{\in}\mathbf{x} \wedge \mathbf{z}{\in}\mathbf{y})$, whence $\mathbf{x}{\cap}\mathbf{y}{\equiv}O$ by lemma 7(ix) and theorem 2(i). Hence $\exists\mathbf{y}(\mathbf{y}{\in}\mathbf{x} \wedge \mathbf{x}{\cap}\mathbf{y}{\equiv}O)$.

2 Suppose instead that $\mathbf{z}{\in}\mathbf{x}{\to}\neg U\mathbf{z}$. It follows by the definition of U and axiom 1(i) that $\mathbf{z}{\in}\mathbf{x}{\to}M\mathbf{z}$. Hence by lemma 5(i) and the definition of S, $\mathbf{z}{\in}\mathbf{x}{\to}S\mathbf{z}$. It follows from the hypothesis $M\mathbf{x}$ that $m\mathbf{y}\ \mathbf{y}{\in}\mathbf{x}$ by the definition of M, whence $\exists\mathbf{y}(S\mathbf{y} \wedge \mathbf{y}{\in}\mathbf{x})$. Hence by theorem 17(i), $\exists\mathbf{y}(\mathsf{E}!V^*(\mathbf{y}) \wedge \mathbf{y}{\in}\mathbf{x})$, whence $\exists\mathbf{u}\exists\mathbf{y}(\mathbf{u}{=}V^*(\mathbf{y}) \wedge \mathbf{y}{\in}\mathbf{x})$ by the definition of $V^*(\mathbf{y})$. Hence by theorem 15(ii), there is a unique level $\mathbf{v_1}$ such that for some $\mathbf{y}$, $\mathbf{v_1}{=}V^*(\mathbf{y}) \wedge \mathbf{y}{\in}\mathbf{x}$, and $\neg\exists\mathbf{w}(\mathbf{w}{\in}\mathbf{v_1} \wedge \exists\mathbf{z}(\mathbf{w}{=}V^*(\mathbf{z}) \wedge \mathbf{z}{\in}\mathbf{x}))$.

3 For a reductio suppose that $\mathbf{z_1}{\in}\mathbf{x} \wedge \mathbf{z_1}{\in}\mathbf{y}$ for some $\mathbf{z_1}$. Since $\mathbf{z}{\in}\mathbf{x}{\to}S\mathbf{z}$, it follows that $S\mathbf{z_1}$, whence $\mathsf{E}!V^*(\mathbf{z_1})$ by theorem 17(i). Since $\mathbf{y}{\subseteq}V^*(\mathbf{y})$ by the definition of $V^*(\mathbf{y})$, it follows by the definition of $\subseteq$ that $\mathbf{z_1}{\in}V^*(\mathbf{y})$. Hence $V^*(\mathbf{z_1}){\in}V^*(\mathbf{y})$ by the definition of $V^*(\mathbf{y})$, lemma 8(iii), theorems 17(iii) and 17(iv), whence $V^*(\mathbf{z_1}){\in}\mathbf{v_1}$. But $V^*(\mathbf{z_1}){\in}\mathbf{v_1}$ and $\mathbf{z_1}{\in}\mathbf{x}$ are together contrary to $\neg\exists\mathbf{w}(\mathbf{w}{\in}\mathbf{v_1} \wedge \exists\mathbf{z}(\mathbf{w}{=}V^*(\mathbf{z}) \wedge \mathbf{z}{\in}\mathbf{x}))$ by the definition of $V^*(\mathbf{z})$. Hence $\neg\exists\mathbf{z}(\mathbf{z}{\in}\mathbf{x} \wedge \mathbf{z}{\in}\mathbf{y})$, whence $\mathbf{x}{\cap}\mathbf{y}{\equiv}O$ by lemma 7(ix) and theorem 2(i). Hence $\exists\mathbf{y}(\mathbf{y}{\in}\mathbf{x} \wedge \mathbf{x}{\cap}\mathbf{y}{\equiv}O)$.

THEOREM 19. *Generalized Intersection*

(i) $\exists\mathbf{x}(\forall\mathbf{y}(A(\mathbf{y}){\to}\mathbf{x}{\in}\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z})) \leftrightarrow \mathsf{E}!(\cap\mathbf{x}A(\mathbf{x}))$

(ii) *Let* $\exists_1\mathbf{x}(\forall\mathbf{y}(A(\mathbf{y}){\to}\mathbf{x}{\in}\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z}))$, *then* $E(\cap\mathbf{x}A(\mathbf{x}))$.

(iii) *Let* $m\mathbf{x}(\forall\mathbf{y}(A(\mathbf{y}){\to}\mathbf{x}{\in}\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z}))$, *then* $M(\cap\mathbf{x}A(\mathbf{x}))$.

(iv) *Let* $m\mathbf{x}(\forall\mathbf{y}(A(\mathbf{y}){\to}\mathbf{x}{\in}\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge S\mathbf{z}))$, *then* $S(\cap\mathbf{x}A(\mathbf{x}))$.

PROOF OF (i). By lemma 5(i) $\exists\mathbf{x}(\forall\mathbf{y}(A(\mathbf{y}){\to}\mathbf{x}{\in}\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z})) \leftrightarrow \exists\mathbf{x}(E\mathbf{x} \wedge \forall\mathbf{y}(A(\mathbf{y}){\to}\mathbf{x}{\in}\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z}))$, whence $\exists\mathbf{x}(\forall\mathbf{y}(A(\mathbf{y}){\to}\mathbf{x}{\in}\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z})) \leftrightarrow \mathsf{E}!(\mathbf{x}{:}\forall\mathbf{y}(A(\mathbf{y}){\to}\mathbf{x}{\in}\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z}))$ by lemma 2(iv). Hence $\exists\mathbf{x}(\forall\mathbf{y}(A(\mathbf{y})\to\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z})) \leftrightarrow \mathsf{E}!(\cap\mathbf{x}A(\mathbf{x}))$ by the definition of $\cap\mathbf{x}A(\mathbf{x})$.

PROOF OF (ii). From the hypothesis it follows that $\exists_1\mathbf{x}(E\mathbf{x} \wedge \forall\mathbf{y}(A(\mathbf{y})\to\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z}))$ by lemma 5(i), whence $\mathbf{x}{:}(\forall\mathbf{y}(A(\mathbf{y})\to\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z})) = \imath\mathbf{z}(E\mathbf{z} \wedge \forall\mathbf{y}(A(\mathbf{y})\to\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z}))$ by lemma 2(iii). Hence $E(\mathbf{x}{:}\forall\mathbf{y}(A(\mathbf{y})\to\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z}))$, whence $E(\cap\mathbf{x}A(\mathbf{x}))$ by the definition of $\cap\mathbf{x}A(\mathbf{x})$.

PROOF OF (iii). From the hypothesis it follows that $m\mathbf{x}(E\mathbf{x} \wedge \forall\mathbf{y}(A(\mathbf{y})\to\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z}))$ by lemma 5(i), whence $\mathbf{x}{:}(\forall\mathbf{y}(A(\mathbf{y})\to\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z})) = \mathbf{x}\cdot(\forall\mathbf{y}(A(\mathbf{y})\to\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z}))$ by lemma 2(ii). Hence $M(\mathbf{x}{:}\forall\mathbf{y}(A(\mathbf{y})\to\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z}))$ by lemma 4, whence $M(\cap\mathbf{x}A(\mathbf{x}))$ by the definition of $\cap\mathbf{x}A(\mathbf{x})$.

PROOF OF (iv).

1 By hypothesis $S\mathbf{z}_1 \wedge A(\mathbf{z}_1)$ for some $\mathbf{z}_1$, whence $\forall\mathbf{y}(A(\mathbf{y})\to\mathbf{x}\in\mathbf{y})\to\mathbf{x}\in\mathbf{z}_1$. Since by hypothesis $m\mathbf{x}(\forall\mathbf{y}(A(\mathbf{y})\to\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge S\mathbf{z}))$, it follows that $m\mathbf{x}(\mathbf{x}\in\mathbf{z}_1 \wedge \forall\mathbf{y}(A(\mathbf{y})\to\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge S\mathbf{z}))$, whence $S(\mathbf{x}\cdot\mathbf{x}\in\mathbf{z}_1 \wedge \forall\mathbf{y}(A(\mathbf{y})\to\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge S\mathbf{z}))$ by theorem 1(i). Since $(\mathbf{x}\in\mathbf{z}_1 \wedge \forall\mathbf{y}(A(\mathbf{y})\to\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge S\mathbf{z})) \leftrightarrow (\forall\mathbf{y}(A(\mathbf{y})\to\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge S\mathbf{z}))$, it follows that $S(\mathbf{x}\cdot\forall\mathbf{y}(A(\mathbf{y})\to\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge S\mathbf{z}))$ by lemma 1(i).

2 Since $(\forall\mathbf{y}(A(\mathbf{y})\to\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge S\mathbf{z})) \leftrightarrow (E\mathbf{x} \wedge \forall\mathbf{y}(A(\mathbf{y})\to\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge S\mathbf{z}))$ by lemma 5(i), it follows that $m\mathbf{x}(E\mathbf{x} \wedge \forall\mathbf{y}(A(\mathbf{y})\to\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge S\mathbf{z}))$, whence $S(\mathbf{z}{:}\forall\mathbf{y}(A(\mathbf{y})\to\mathbf{x}\in\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge S\mathbf{z}))$ by lemma 2(ii). Hence $S(\cap\mathbf{x}A(\mathbf{x}))$ by the definition of $\cap\mathbf{x}A(\mathbf{x})$.

COROLLARY. *Let* $m\mathbf{z}\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\to\mathbf{z}\in\mathbf{y}) \wedge \exists\mathbf{z}\ \mathbf{z}\in\mathbf{x}$, *then* $S\cap\mathbf{x}$.

PROOF. From the hypothesis it follows that $\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\to M\mathbf{y}) \wedge \mathsf{E}!\mathbf{x}$ by the definition of M and axiom 1(i), whence $\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\to S\mathbf{y})$ by the definitions of S and E. Hence $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x} \wedge S\mathbf{z})$, whence $S\cap\mathbf{x}$ by theorem 19(iv) and the definition of $\cap\mathbf{x}$.

THEOREM 20. *Union*

(i) $M\mathbf{x} \wedge M\mathbf{y} \leftrightarrow M(\mathbf{x}\cup\mathbf{y})$

(ii) $S\mathbf{x} \wedge S\mathbf{y} \leftrightarrow S(\mathbf{x}\cup\mathbf{y})$

PROOF OF (i).

1 For the $\to$ half, suppose $M\mathbf{x} \wedge M\mathbf{y}$. Then $m\mathbf{z}\ \mathbf{z}\in\mathbf{x}$ by the definition of M; a fortiori $m\mathbf{z}(\mathbf{z}\in\mathbf{x} \vee \mathbf{z}\in\mathbf{y})$. Hence $m\mathbf{z}(E\mathbf{z} \wedge M\mathbf{x} \wedge M\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}\in\mathbf{y}))$ by lemma 5(i), whence $\mathsf{E}!(\mathbf{z}\cdot M\mathbf{x} \wedge M\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}\in\mathbf{y}))$ by lemma 2(i). Hence $M(\mathbf{z}\cdot M\mathbf{x} \wedge M\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}\in\mathbf{y}))$ by the definition of $\mathbf{z}\cdot(M\mathbf{x} \wedge M\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}\in\mathbf{y}))$, whence $M(\mathbf{x}\cup\mathbf{y})$ by the definition of $\cup$.

2 For the $\leftarrow$ half, suppose $M(\mathbf{x}\cup\mathbf{y})$. Then $\mathsf{E}!(\mathbf{x}\cup\mathbf{y})$ by lemma 3(i). Hence $m\mathbf{z}(E\mathbf{z} \wedge M\mathbf{x} \wedge M\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}\in\mathbf{y}))$ by lemma 2(i) and the definition of $\mathbf{x}\cup\mathbf{y}$, whence $M\mathbf{x} \wedge M\mathbf{y}$.

PROOF OF (ii).

1 For the $\rightarrow$ half, suppose $S\mathbf{x} \wedge S\mathbf{y}$. With a view to using theorem 1(iii) we shall prove (i) $m\mathbf{z}(E\mathbf{z} \wedge M\mathbf{x} \wedge M\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}\in\mathbf{y}))$ and (ii) $\exists\mathbf{u}\forall\mathbf{z}((E\mathbf{z} \wedge M\mathbf{x} \wedge M\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}\in\mathbf{y}))\rightarrow\mathbf{z}\in\mathbf{u})$.

2 For (i), since $S\mathbf{x}$, it follows that $m\mathbf{z}\ \mathbf{z}\in\mathbf{x}$ by the definitions of S and M; a fortiori $m\mathbf{z}(\mathbf{z}\in\mathbf{x} \vee \mathbf{z}\in\mathbf{y})$. Since $S\mathbf{x}$ and $S\mathbf{y}$, it follows that $M\mathbf{x}$ and $M\mathbf{y}$ by the definition of M. Hence $m\mathbf{z}(E\mathbf{z} \wedge M\mathbf{x} \wedge M\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}\in\mathbf{y}))$ by lemma 5(i).

3 For (ii), since $S\mathbf{x}$ and $S\mathbf{y}$, it follows by axiom 2(iv) that $\mathbf{x}\subseteq\mathbf{v}$ and $\mathbf{y}\subseteq\mathbf{w}$ for some levels $\mathbf{v}$ and $\mathbf{w}$. By theorem 14, $\mathbf{v}\in\mathbf{w} \vee \mathbf{v}=\mathbf{w} \vee \mathbf{w}\in\mathbf{v}$. Suppose $\mathbf{v}\in\mathbf{w}$, then from $\mathbf{x}\subseteq\mathbf{v}$ it follows that $\mathbf{x}\in\mathbf{w}$ by theorem 9, whence $\mathbf{x}\subseteq\mathbf{w}$ by the corollary of theorem 8. Since $\mathbf{y}\subseteq\mathbf{w}$ too, it follows that $\forall\mathbf{z}((\mathbf{z}\in\mathbf{x} \vee \mathbf{z}\in\mathbf{y})\rightarrow\mathbf{z}\in\mathbf{w})$ by the definition of $\subseteq$, whence $\exists\mathbf{u}\forall\mathbf{z}((\mathbf{z}\in\mathbf{x} \vee \mathbf{z}\in\mathbf{y})\rightarrow\mathbf{z}\in\mathbf{u})$. Suppose $\mathbf{w}\in\mathbf{v}$, then by similar reasoning $\mathbf{x}\subseteq\mathbf{v}$ and $\mathbf{y}\subseteq\mathbf{v}$, whence $\exists\mathbf{u}\forall\mathbf{z}((\mathbf{z}\in\mathbf{x} \vee \mathbf{z}\in\mathbf{y})\rightarrow\mathbf{z}\in\mathbf{u})$. Suppose $\mathbf{v}=\mathbf{w}$, then again both $\mathbf{x}\subseteq\mathbf{v}$ and $\mathbf{y}\subseteq\mathbf{v}$, whence $\exists\mathbf{u}\forall\mathbf{z}((\mathbf{z}\in\mathbf{x} \vee \mathbf{z}\in\mathbf{y})\rightarrow\mathbf{z}\in\mathbf{u})$. In each case, then, $\exists\mathbf{u}\forall\mathbf{z}((\mathbf{z}\in\mathbf{x} \vee \mathbf{z}\in\mathbf{y})\rightarrow\mathbf{z}\in\mathbf{u})$, whence $\exists\mathbf{u}\forall\mathbf{z}((E\mathbf{z} \wedge M\mathbf{x} \wedge M\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}\in\mathbf{y}))\rightarrow\mathbf{z}\in\mathbf{u})$ by lemma 5(i).

4 From (i) and (ii), it follows that $S(\mathbf{z}{\cdot}M\mathbf{x} \wedge M\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}\in\mathbf{y}))$ by theorem 1(iii), whence $S(\mathbf{x}\cup\mathbf{y})$ by the definition of $\cup$.

5 For the $\leftarrow$ half, suppose $S(\mathbf{x}\cup\mathbf{y})$. Then $M(\mathbf{x}\cup\mathbf{y})$ by the definition of S, whence $M\mathbf{x} \wedge M\mathbf{y}$ by theorem 20(i), and also $\mathbf{x}\cup\mathbf{y}=\mathbf{x}\cup\mathbf{y}$ by lemma 3(i). Hence $\forall\mathbf{z}(\mathbf{z}\in\mathbf{x}\rightarrow\mathbf{z}\in\mathbf{x}\cup\mathbf{y})$ and $\forall\mathbf{z}(\mathbf{z}\in\mathbf{y}\rightarrow\mathbf{z}\in\mathbf{x}\cup\mathbf{y})$ by lemmas 4 and 5(i), and the definition of $\cup$, whence $\mathbf{x}\subseteq\mathbf{x}\cup\mathbf{y}$ and $\mathbf{y}\subseteq\mathbf{x}\cup\mathbf{y}$ by the definition of $\subseteq$. Hence $S\mathbf{x}$ and $S\mathbf{y}$ by axiom 2(iii).

THEOREM 21. *Generalized union*

(i) $(\forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z})) \leftrightarrow M(\cup\mathbf{x}A(\mathbf{x}))$

(ii) $(\forall\mathbf{y}(A(\mathbf{y})\rightarrow S\mathbf{y}) \wedge S(\mathbf{z}{:}A(\mathbf{z}))) \leftrightarrow S(\cup\mathbf{x}A(\mathbf{x}))$

PROOF OF (i).

1 For the $\rightarrow$ half, suppose $\forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z})$. Then $\exists\mathbf{z}(A(\mathbf{z}) \wedge M\mathbf{z})$, whence $\exists\mathbf{z}(A(\mathbf{z}) \wedge m\mathbf{z_1}\ \mathbf{z_1}\in\mathbf{z})$ by the definition of M. Hence $m\mathbf{z_1}(E\mathbf{z_1} \wedge \forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge \mathbf{z_1}\in\mathbf{z}))$ by lemma 5(i), whence $\mathsf{E}!(\mathbf{x}{\cdot}\forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge \mathbf{x}\in\mathbf{z}))$ by lemma 2(i). Hence $M(\mathbf{x}{\cdot}\forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge \mathbf{x}\in\mathbf{z}))$ by the definition of $\mathbf{x}{\cdot}(\forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge \mathbf{x}\in\mathbf{z}))$, whence $M(\cup\mathbf{x}A(\mathbf{x}))$ by the definition of $\cup\mathbf{x}A(\mathbf{x})$.

2 For the $\leftarrow$ half, suppose $M(\cup\mathbf{x}A(\mathbf{x}))$. Then $\mathsf{E}!(\cup\mathbf{x}A(\mathbf{x}))$ by lemma 3(i). Hence $m\mathbf{z_1}(E\mathbf{z_1} \wedge \forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge \mathbf{z_1}\in\mathbf{z}))$ by lemma 2(i) and the definition of $\cup\mathbf{x}A(\mathbf{x})$, whence $\forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z})$.

PROOF OF (ii).

1 For the $\rightarrow$ half, suppose $\forall\mathbf{y}(A(\mathbf{y})\rightarrow S\mathbf{y}) \wedge S(\mathbf{z}{:}A(\mathbf{z}))$. Then $M(\mathbf{z}{:}A(\mathbf{z}))$ by the definition of S. Hence $\mathsf{E}!(\mathbf{z}{:}A(\mathbf{z}))$ by lemma 3(i), whence $\exists_1\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z})) \vee m\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$ by lemma 2(iv).

2 Suppose $\exists_1\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$. Then $\mathbf{z}{:}A(\mathbf{z}) = \iota\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$ by lemma 2(iii), whence $\mathbf{x}\in\mathbf{z}{:}A(\mathbf{z}) \leftrightarrow \mathbf{x}\in\iota\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$. Hence $\mathbf{x}{:}(\mathbf{x}\in\mathbf{z}{:}A(\mathbf{z})) \equiv \mathbf{x}{:}(\mathbf{x}\in\iota\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z})))$ by lemma 1(ii). Since $M(\mathbf{z}{:}A(\mathbf{z}))$, it follows by lemma 5(iii) that $\mathbf{z}{:}A(\mathbf{z}) = \mathbf{x}{:}(\mathbf{x}\in\iota\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z})))$. From the hypothesis $\forall\mathbf{y}(A(\mathbf{y})\rightarrow S\mathbf{y})$ and the definition of S, it follows that $\mathbf{x}\in\iota\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z})) \leftrightarrow (\forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge \mathbf{x}\in\mathbf{z}))$, whence $\mathbf{z}{:}A(\mathbf{z}) = \mathbf{x}{:}(\forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge \mathbf{x}\in\mathbf{z}))$ by lemma 1(ii). Hence $S(\mathbf{x}{:}\forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge \mathbf{x}\in\mathbf{z}))$. By the

definition of M and lemma 5(i), $m\mathbf{x}(E\mathbf{x} \wedge \forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge \mathbf{x}\in\mathbf{z}))$, whence $S(\mathbf{x}\cdot\forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge \mathbf{x}\in\mathbf{z}))$ by lemma 2(ii). Hence $S(\cup\mathbf{x}A(\mathbf{x}))$ by the definition of $\cup\mathbf{x}A(\mathbf{x})$.

3 Suppose instead $m\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$. With a view to using theorem 1(iii) we shall prove (i) $m\mathbf{z}_1(E\mathbf{z}_1 \wedge \forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge \mathbf{z}_1\in\mathbf{z}))$ and (ii) $\exists\mathbf{u}\forall\mathbf{z}_1((E\mathbf{z}_1 \wedge \forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge \mathbf{z}_1\in\mathbf{z})) \rightarrow \mathbf{z}_1\in\mathbf{u})$.

4 For (i), since $m\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$ it follows that $A(\mathbf{z}_2)$ for some $\mathbf{z}_2$. From the hypothesis $\forall\mathbf{y}(A(\mathbf{y})\rightarrow S\mathbf{y})$ it follows that $\forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge M\mathbf{z}_2$ by the definition of S, whence $m\mathbf{z}_1(\forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \mathbf{z}_1\in\mathbf{z}_2)$ by the definition of M. Since $\forall\mathbf{y}(\mathbf{y}\in\mathbf{z}_2\rightarrow\exists\mathbf{z}(A(\mathbf{z}) \wedge \mathbf{y}\in\mathbf{z}))$, it follows that $m\mathbf{z}_1(E\mathbf{z}_1 \wedge \forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge \mathbf{z}_1\in\mathbf{z}))$ by lemma 5(i).

5 For (ii), since $m\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$, it follows that $\mathbf{z}{:}A(\mathbf{z}) = \mathbf{z}\cdot A(\mathbf{z})$ by lemma 2(ii), whence $S(\mathbf{z}\cdot A(\mathbf{z}))$. Hence $\forall\mathbf{y}((E\mathbf{y} \wedge A(\mathbf{y})) \rightarrow \mathbf{y}\in\mathbf{v})$ for some level $\mathbf{v}$ by theorem 1(iii). From the hypothesis $\forall\mathbf{y}(A(\mathbf{y})\rightarrow S\mathbf{y})$ it follows that $\forall\mathbf{y}(A(\mathbf{y})\rightarrow(M\mathbf{y} \wedge E\mathbf{y}))$ by the definition of S, whence $\forall\mathbf{x}(\exists\mathbf{y}(A(\mathbf{y}) \wedge \mathbf{x}\in\mathbf{y}) \rightarrow \exists\mathbf{y}(\mathbf{y}\in\mathbf{v} \wedge \mathbf{x}\in\mathbf{y}))$. Hence $\forall\mathbf{x}(\exists\mathbf{y}(A(\mathbf{y}) \wedge \mathbf{x}\in\mathbf{y}) \rightarrow \mathbf{x}\in\mathbf{v})$ by theorem 8. Hence $\exists\mathbf{u}\forall\mathbf{x}((E\mathbf{x} \wedge \forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{y}(A(\mathbf{y}) \wedge \mathbf{x}\in\mathbf{y})) \rightarrow \mathbf{x}\in\mathbf{u})$ by lemma 5(i).

6 From (i) and (ii) it follows that $S(\mathbf{x}\cdot\forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{z}(A(\mathbf{z}) \wedge \mathbf{x}\in\mathbf{z}))$ by theorem 1(iii), whence $S(\cup\mathbf{x}A(\mathbf{x}))$ by the definition of $\cup\mathbf{x}A(\mathbf{x})$.

7 For the $\leftarrow$ half, suppose $S(\cup\mathbf{x}A(\mathbf{x}))$. Then $M(\cup\mathbf{x}A(\mathbf{x}))$ by the definition of S, whence $\forall\mathbf{y}(A(\mathbf{y})\rightarrow M\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z})$ by theorem 21(i). Consider an arbitrary $\mathbf{z}_1$ such that $A(\mathbf{z}_1)$. Then $M\mathbf{z}_1$. Also $\forall\mathbf{z}_2(\mathbf{z}_2\in\mathbf{z}_1\rightarrow\mathbf{z}_2\in\cup\mathbf{x}A(\mathbf{x}))$ by lemmas 3(i), 4 and 5(i), and the definition of $\cup$. By axiom 2(iv), $\forall\mathbf{z}_2(\mathbf{z}_2\in\cup\mathbf{x}A(\mathbf{x})\rightarrow\mathbf{z}_2\in\mathbf{v})$ for some level $\mathbf{v}$, whence $\forall\mathbf{z}_2(\mathbf{z}_2\in\mathbf{z}_1\rightarrow\mathbf{z}_2\in\mathbf{v})$. Hence $\mathbf{z}_1\subseteq\mathbf{v}$ by the definition of $\subseteq$, whence $S\mathbf{z}_1$ by theorem 1(ii). Since $\mathbf{z}_1$ is arbitrary, we can generalize to get $\forall\mathbf{y}(A(\mathbf{y})\rightarrow S\mathbf{y})$.

8 Since $\exists\mathbf{z}A(\mathbf{z})$ and $\forall\mathbf{y}(A(\mathbf{y})\rightarrow S\mathbf{y})$, it follows that $\exists\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$ by the definition of S, whence $\mathbf{z}{:}A(\mathbf{z}) = \imath\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$ or $\mathbf{z}{:}A(\mathbf{z}) = \mathbf{z}\cdot A(\mathbf{z})$ by lemmas 2(iii) and 2(iv). Suppose $\mathbf{z}{:}A(\mathbf{z}) = \imath\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$. Then $A(\mathbf{z}{:}A(\mathbf{z}))$, whence $S(\mathbf{z}{:}A(\mathbf{z}))$. Suppose instead $\mathbf{z}{:}A(\mathbf{z}) = \mathbf{z}\cdot A(\mathbf{z})$. Then $M(\mathbf{z}{:}A(\mathbf{z}))$ and $\mathbf{z}_1\in\mathbf{z}{:}A(\mathbf{z})\rightarrow A(\mathbf{z}_1)$ by lemma 4, whence by the reasoning in step 7 $\mathbf{z}_1\in\mathbf{z}{:}A(\mathbf{z})\rightarrow\mathbf{z}_1\subseteq\mathbf{v}$. By axiom 2(v), $\mathbf{v}\in\mathbf{u}$ for some level $\mathbf{u}$, whence $\mathbf{z}_1\in\mathbf{z}{:}A(\mathbf{z})\rightarrow\mathbf{z}_1\in\mathbf{u}$ by theorem 9. Hence $\mathbf{z}{:}A(\mathbf{z})\subseteq\mathbf{u}$ by the definition of $\subseteq$, whence $S(\mathbf{z}{:}A(\mathbf{z}))$ by theorem 1(ii).

COROLLARIES. (i) $(\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow S\mathbf{y}) \wedge M\mathbf{x}) \leftrightarrow M\cup\mathbf{x}$

(ii) $(\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow S\mathbf{y}) \wedge S\mathbf{x}) \leftrightarrow S\cup\mathbf{x}$

PROOF OF (i). By the definitions of S and M, lemma 5(i) and axiom 1(iv), $(\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow S\mathbf{y}) \wedge M\mathbf{x}) \leftrightarrow (\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow M\mathbf{y}) \wedge \exists\mathbf{z}\,\mathbf{z}\in\mathbf{x})$, whence $(\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow S\mathbf{y}) \wedge M\mathbf{x}) \leftrightarrow M\cup\mathbf{x}$ by theorem 21(i) and the definition of $\cup\mathbf{x}$.

PROOF OF (ii). By the definition of S, lemmas 2(iv), 3(i) and 5(iii), and axiom 1(iv), $(\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow S\mathbf{y}) \wedge S\mathbf{x}) \leftrightarrow ((\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow S\mathbf{y}) \wedge S(\mathbf{z}{:}\mathbf{z}\in\mathbf{x}))$, whence $(\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow S\mathbf{y}) \wedge S\mathbf{x}) \leftrightarrow S\cup\mathbf{x}$ by theorem 21(ii) and the definition of $\cup\mathbf{x}$.

THEOREM 22. *Sets and levels II*

(i) $S\mathbf{x}\leftrightarrow(M\mathbf{x} \wedge \exists\mathbf{u}\ \mathbf{x}\in\mathbf{u})$

(ii) $S\mathbf{x}\leftrightarrow(M\mathbf{x} \wedge \mathsf{E}!V^\dagger(\mathbf{x}))$

PROOF OF (i).

1 For the $\rightarrow$ half, suppose $S\mathbf{x}$. Then $M\mathbf{x}$ by the definition of S. Also by axiom 2(iv) $\mathbf{x}\subseteq\mathbf{u_1}$ for some level $\mathbf{u_1}$, whence by axiom 2(v) $\mathbf{u_1}\in\mathbf{u_2}$ for some level $\mathbf{u_2}$. Since $\mathbf{x}\subseteq\mathbf{u_1}$ and $\mathbf{u_1}\in\mathbf{u_2}$, it follows that $\mathbf{x}\in\mathbf{u_2}$ by theorem 9, whence $\exists\mathbf{u}\ \mathbf{x}\in\mathbf{u}$.

2 For the $\leftarrow$ half, suppose $M\mathbf{x} \wedge \exists\mathbf{u}\ \mathbf{x}\in\mathbf{u}$. Then $S\mathbf{x}$ by the definitions of E and S.

PROOF OF (ii).

1 For the $\rightarrow$ half, suppose $S\mathbf{x}$. Then $M\mathbf{x} \wedge \exists\mathbf{u}\ \mathbf{x}\in\mathbf{u}$ by theorem 22(i). Hence $\exists_1\mathbf{v}(\mathbf{x}\in\mathbf{v} \wedge \neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{v} \wedge \mathbf{x}\in\mathbf{w}))$ by theorem 15(ii), whence $\mathsf{E}!V^\dagger(\mathbf{x})$ by the definition of $V^\dagger(\mathbf{x})$.

2 For the $\leftarrow$ half, suppose $M\mathbf{x} \wedge \mathsf{E}!V^\dagger(\mathbf{x})$. Then by the definition of $V^\dagger(\mathbf{x})$ it follows that $\mathbf{x}\in V^\dagger(\mathbf{x})$, whence $S\mathbf{x}$ by the definitions of E and S.

THEOREM 23. *Pairing*

(i) $(E\mathbf{x} \wedge E\mathbf{y} \wedge \mathbf{x}\neq\mathbf{y})\leftrightarrow\mathsf{E}!|\mathbf{x}, \mathbf{y}|$

(ii) $\mathsf{E}!|\mathbf{x}, \mathbf{y}|\leftrightarrow S|\mathbf{x}, \mathbf{y}|$

PROOF OF (i). By lemma 2(i) and the definition of $|\mathbf{x}, \mathbf{y}|$, $m\mathbf{z}(E\mathbf{z} \wedge (\mathbf{z}=\mathbf{x} \vee \mathbf{z}=\mathbf{y})) \leftrightarrow \mathsf{E}!|\mathbf{x}, \mathbf{y}|$. Hence $E\mathbf{x} \wedge E\mathbf{y} \wedge \mathbf{x}\neq\mathbf{y} \leftrightarrow \mathsf{E}!|\mathbf{x}, \mathbf{y}|$ by lemmas 3(i), 3(ii) and 3(iii).

PROOF OF (ii).

1 For the $\rightarrow$ half, suppose $\mathsf{E}!|\mathbf{x}, \mathbf{y}|$. Then $m\mathbf{z}(E\mathbf{z} \wedge (\mathbf{z}=\mathbf{x} \vee \mathbf{z}=\mathbf{y}))$ by lemma 2(i) and the definition of $|\mathbf{x}, \mathbf{y}|$, whence $E\mathbf{x} \wedge E\mathbf{y} \wedge \mathbf{x}\neq\mathbf{y}$. Hence by lemma 3(iii) it follows that either (i) $U\mathbf{x} \wedge U\mathbf{y}$ or (ii) $S\mathbf{x} \wedge S\mathbf{y}$ or (iii) $S\mathbf{x} \wedge U\mathbf{y}$ or (iv) $U\mathbf{x} \wedge S\mathbf{y}$. With a view to using theorem 1(iii) we prove $\exists\mathbf{u}\forall\mathbf{z}((E\mathbf{z} \wedge (\mathbf{z}=\mathbf{x} \vee \mathbf{z}=\mathbf{y}))\rightarrow\mathbf{z}\in\mathbf{u})$ for each case.

2 For case (i), from $U\mathbf{x} \wedge U\mathbf{y} \wedge \mathbf{x}\neq\mathbf{y}$ it follows that $m\mathbf{z_1}(E\mathbf{z_1} \wedge U\mathbf{z_1})$. Hence by lemma 2(i) $\mathsf{E}!(\mathbf{z_1}\cdot U\mathbf{z_1})$, whence $V_1 = \mathbf{z_1}\cdot U\mathbf{z_1}$ by lemma 9(i). Since $(\mathbf{z}=\mathbf{x} \vee \mathbf{z}=\mathbf{y})\rightarrow U\mathbf{z}$, it follows that $(\mathbf{z}=\mathbf{x} \vee \mathbf{z}=\mathbf{y})\rightarrow\mathbf{z}\in V_1$ by lemma 9(ii). So $\exists\mathbf{u}\forall\mathbf{z}((E\mathbf{z} \wedge (\mathbf{z}=\mathbf{x} \vee \mathbf{z}=\mathbf{y}))\rightarrow\mathbf{z}\in\mathbf{u})$ by the definition of V.

3 For case (ii), from $S\mathbf{x} \wedge S\mathbf{y}$ it follows by theorem 22(i) that $\mathbf{x}\in\mathbf{v}$ and $\mathbf{y}\in\mathbf{w}$ for some levels $\mathbf{v}$, $\mathbf{w}$. By theorem 14, $\mathbf{v}\in\mathbf{w} \vee \mathbf{v}=\mathbf{w} \vee \mathbf{w}\in\mathbf{v}$. Suppose $\mathbf{v}\in\mathbf{w}$, then by theorem 8 $\mathbf{x}\in\mathbf{w}$. Also $\mathbf{y}\in\mathbf{w}$, so $\exists\mathbf{u}\forall\mathbf{z}((E\mathbf{z} \wedge (\mathbf{z}=\mathbf{x} \vee \mathbf{z}=\mathbf{y}))\rightarrow\mathbf{z}\in\mathbf{u})$. Suppose $\mathbf{v}=\mathbf{w}$, then $\mathbf{x}\in\mathbf{v}$ and $\mathbf{y}\in\mathbf{v}$, whence $\exists\mathbf{u}\forall\mathbf{z}((E\mathbf{z} \wedge (\mathbf{z}=\mathbf{x} \vee \mathbf{z}=\mathbf{y}))\rightarrow\mathbf{z}\in\mathbf{u})$. Suppose $\mathbf{w}\in\mathbf{v}$, then by theorem 8, $\mathbf{y}\in\mathbf{v}$. Also $\mathbf{x}\in\mathbf{v}$, so $\exists\mathbf{u}\forall\mathbf{z}((E\mathbf{z} \wedge (\mathbf{z}=\mathbf{x} \vee \mathbf{z}=\mathbf{y}))\rightarrow\mathbf{z}\in\mathbf{u})$. From $\mathbf{v}\in\mathbf{w} \vee \mathbf{v}=\mathbf{w} \vee \mathbf{w}\in\mathbf{v}$, then, it follows that $\exists\mathbf{u}\forall\mathbf{z}((E\mathbf{z} \wedge (\mathbf{z}=\mathbf{x} \vee \mathbf{z}=\mathbf{y}))\rightarrow\mathbf{z}\in\mathbf{u})$.

4 For case (iii), from $S\mathbf{x}$, it follows by theorem 22(i) that $\mathbf{x}\in\mathbf{v}$ for some level $\mathbf{v}$. By lemma 8(iv) it follows from $U\mathbf{y}$ that $\mathbf{y}\in\mathbf{v}$. Hence $\exists\mathbf{u}\forall\mathbf{z}((E\mathbf{z} \wedge (\mathbf{z}=\mathbf{x} \vee \mathbf{z}=\mathbf{y}))\rightarrow\mathbf{z}\in\mathbf{u})$.

5 For case (iv), $\exists\mathbf{u}\forall\mathbf{z}((E\mathbf{z} \wedge (\mathbf{z}=\mathbf{x} \vee \mathbf{z}=\mathbf{y}))\rightarrow\mathbf{z}\in\mathbf{u})$ is proved by the reasoning in step 4.

6 Since $m\mathbf{z}(E\mathbf{z} \wedge (\mathbf{z}=\mathbf{x} \vee \mathbf{z}=\mathbf{y}))$ and $\exists\mathbf{u}\forall\mathbf{z}((E\mathbf{z} \wedge (\mathbf{z}=\mathbf{x} \vee \mathbf{z}=\mathbf{y}))\rightarrow\mathbf{z}\in\mathbf{u})$, it follows by theorem 1(iii) that $S(\mathbf{z}\cdot\mathbf{z}=\mathbf{x} \vee \mathbf{z}=\mathbf{y})$, whence $S|\mathbf{x}, \mathbf{y}|$ by the definition of $|\mathbf{x}, \mathbf{y}|$.

7 For the $\leftarrow$ half, suppose $S|\mathbf{x}, \mathbf{y}|$. Then $\mathsf{E}!|\mathbf{x}, \mathbf{y}|$ by lemmas 3(i) and 3(ii).

THEOREM 24. *Adjunction*

(i) $M\mathbf{x} \wedge E\mathbf{y} \leftrightarrow M(\mathbf{x}\oplus\mathbf{y})$

(ii) $S\mathbf{x} \wedge E\mathbf{y} \leftrightarrow S(\mathbf{x}\oplus\mathbf{y})$

PROOF OF (i).

1 For the $\rightarrow$ half, suppose $M\mathbf{x} \wedge E\mathbf{y}$. Then $m\mathbf{z}\ \mathbf{z}\in\mathbf{x}$ by the definition of M; a fortiori $m\mathbf{z}(\mathbf{z}\in\mathbf{x} \vee \mathbf{z}=\mathbf{y})$. Hence $m\mathbf{z}(E\mathbf{z} \wedge M\mathbf{x} \wedge E\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}=\mathbf{y}))$ by lemma 5(i), whence $\mathsf{E}!(\mathbf{z}\cdot M\mathbf{x} \wedge E\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}=\mathbf{y}))$ by lemma 2(i). Hence $M(\mathbf{z}\cdot M\mathbf{x} \wedge E\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}=\mathbf{y}))$ by the definition of $\mathbf{z}\cdot(M\mathbf{x} \wedge E\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}=\mathbf{y}))$, whence $M(\mathbf{x}\oplus\mathbf{y})$ by the definition of $\oplus$.

2 For the $\leftarrow$ half, suppose $M(\mathbf{x}\oplus\mathbf{y})$. Then $\mathsf{E}!\mathbf{x}\oplus\mathbf{y}$ by lemma 3(i). Hence $m\mathbf{z}(E\mathbf{z} \wedge M\mathbf{x} \wedge E\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}=\mathbf{y}))$ by lemma 2(i) and the definition of $\mathbf{x}\oplus\mathbf{y}$, whence $M\mathbf{x} \wedge E\mathbf{y}$.

PROOF OF (ii).

1 For the $\rightarrow$ half, suppose $S\mathbf{x} \wedge E\mathbf{y}$. With a view to using theorem 1(iii) we shall prove (i) $m\mathbf{z}(E\mathbf{z} \wedge M\mathbf{x} \wedge E\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}=\mathbf{y}))$ and (ii) $\exists\mathbf{u}\forall\mathbf{z}((E\mathbf{z} \wedge M\mathbf{x} \wedge E\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}=\mathbf{y}))\rightarrow\mathbf{z}\in\mathbf{u})$.

2 For (i), since $S\mathbf{x}$, it follows that $M\mathbf{x}$ and $m\mathbf{z}\ \mathbf{z}\in\mathbf{x}$ by the definitions of S and M; a fortiori $m\mathbf{z}(\mathbf{z}\in\mathbf{x} \vee \mathbf{z}=\mathbf{y})$. Hence $m\mathbf{z}(E\mathbf{z} \wedge M\mathbf{x} \wedge E\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}=\mathbf{y}))$ by lemma 5(i).

3 For (ii), since $S\mathbf{x} \wedge E\mathbf{y}$, it follows by lemma 3(iii) that $S\mathbf{x} \wedge (U\mathbf{y} \vee S\mathbf{y})$. Then by theorem 22(i), $\mathbf{x}\in\mathbf{v}$ for some level $\mathbf{v}$. Suppose instead $U\mathbf{y}$, then by lemma 8(iv), $\mathbf{y}\in\mathbf{w}$ for some level $\mathbf{w}$. Suppose $S\mathbf{y}$, then by theorem 22(i), $\mathbf{y}\in\mathbf{w}$ for some level $\mathbf{w}$. So either way $\mathbf{y}\in\mathbf{w}$ for some level $\mathbf{w}$. By theorem 14, $\mathbf{v}\in\mathbf{w} \vee \mathbf{v}=\mathbf{w} \vee \mathbf{w}\in\mathbf{v}$. Suppose $\mathbf{v}\in\mathbf{w}$, then by theorem 8, $\mathbf{x}\in\mathbf{w}$, whence $\mathbf{x}\subseteq\mathbf{w}$ by the corollary of theorem 8. Also $\mathbf{y}\in\mathbf{w}$, so $\exists\mathbf{u}\forall\mathbf{z}((\mathbf{z}\in\mathbf{x} \vee \mathbf{z}=\mathbf{y})\rightarrow\mathbf{z}\in\mathbf{u})$ by the definition of $\subseteq$. Suppose $\mathbf{v}=\mathbf{w}$, then $\mathbf{x}\in\mathbf{v}$ and $\mathbf{y}\in\mathbf{v}$, whence $\mathbf{x}\subseteq\mathbf{v}$ by the corollary of theorem 8. Hence $\exists\mathbf{u}\forall\mathbf{z}((\mathbf{z}\in\mathbf{x} \vee \mathbf{z}=\mathbf{y})\rightarrow\mathbf{z}\in\mathbf{u})$ by the definition of $\subseteq$. Suppose $\mathbf{w}\in\mathbf{v}$, then by theorem 8, $\mathbf{y}\in\mathbf{v}$. Also $\mathbf{x}\in\mathbf{v}$, whence $\mathbf{x}\subseteq\mathbf{v}$ by the corollary of theorem 8. So $\exists\mathbf{u}\forall\mathbf{z}((\mathbf{z}\in\mathbf{x} \vee \mathbf{z}=\mathbf{y})\rightarrow\mathbf{z}\in\mathbf{u})$ by the definition of $\subseteq$. From $\mathbf{v}\in\mathbf{w} \vee \mathbf{v}=\mathbf{w} \vee \mathbf{w}\in\mathbf{v}$, then, it follows that $\exists\mathbf{u}\forall\mathbf{z}((\mathbf{z}\in\mathbf{x} \vee \mathbf{z}=\mathbf{y})\rightarrow\mathbf{z}\in\mathbf{u})$, whence $\exists\mathbf{u}\forall\mathbf{z}((E\mathbf{z} \wedge M\mathbf{x} \wedge E\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}=\mathbf{y}))\rightarrow\mathbf{z}\in\mathbf{u})$ by lemma 5(i).

4 Since $m\mathbf{z}(E\mathbf{z} \wedge M\mathbf{x} \wedge E\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}=\mathbf{y}))$ and $\exists\mathbf{u}\forall\mathbf{z}((E\mathbf{z} \wedge M\mathbf{x} \wedge E\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}=\mathbf{y}))\rightarrow\mathbf{z}\in\mathbf{u})$, it follows by theorem 1(iii) that $S(\mathbf{z}\cdot M\mathbf{x} \wedge E\mathbf{y} \wedge (\mathbf{z}\in\mathbf{x} \vee \mathbf{z}=\mathbf{y}))$, whence $S(\mathbf{x}\oplus\mathbf{y})$ by the definition of $\mathbf{x}\oplus\mathbf{y}$.

5 For the $\leftarrow$ half, suppose $S(\mathbf{x}\oplus\mathbf{y})$. Then $M(\mathbf{x}\oplus\mathbf{y})$ by the definition of S, whence $M\mathbf{x} \wedge E\mathbf{y}$ by theorem 24(i), and also $\mathbf{x}\oplus\mathbf{y}=\mathbf{x}\oplus\mathbf{y}$ by lemma 3(i). Hence $\forall\mathbf{z}(\mathbf{z}\in\mathbf{x}\rightarrow\mathbf{z}\in\mathbf{x}\oplus\mathbf{y})$ by lemmas 4 and 5(i) and the definition of $\oplus$, whence $\mathbf{x}\subseteq\mathbf{x}\oplus\mathbf{y}$ by the definition of $\subseteq$. Hence $S\mathbf{x}$ by axiom 2(iii).

THEOREM 25. *Power multitude*

(i) *Let* $\exists\mathbf{y}_1\exists\mathbf{y}_2(\mathbf{y}_1\in\mathbf{x} \wedge \mathbf{y}_2\in\mathbf{x} \wedge \mathbf{y}_1\neq\mathbf{y}_2 \wedge \forall\mathbf{y}_3(\mathbf{y}_3\in\mathbf{x}\rightarrow (\mathbf{y}_3=\mathbf{y}_1 \vee \mathbf{y}_3=\mathbf{y}_2)))$, *then* $\iota\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\subseteq\mathbf{x})=\mathbf{x} \wedge P(\mathbf{x})=\mathbf{x}$.

(ii) *Let* $\exists\mathbf{y}_1\exists\mathbf{y}_2\exists\mathbf{y}_3(\mathbf{y}_1\in\mathbf{x} \wedge \mathbf{y}_2\in\mathbf{x} \wedge \mathbf{y}_3\in\mathbf{x} \wedge \mathbf{y}_1\neq\mathbf{y}_2 \wedge \mathbf{y}_1\neq\mathbf{y}_3 \wedge \mathbf{y}_2\neq\mathbf{y}_3)$, *then* $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\subseteq\mathbf{x}) \wedge P(\mathbf{x})=\mathbf{y}\cdot\mathbf{y}\subseteq\mathbf{x}$.

(iii) $M\mathbf{x} \leftrightarrow M(P(\mathbf{x}))$

(iv) $S\mathbf{x} \leftrightarrow S(P(\mathbf{x}))$

PROOF OF (i).

1 By the hypothesis $\mathbf{z}_1\in\mathbf{x}\wedge\mathbf{z}_2\in\mathbf{x}\wedge\mathbf{z}_1\neq\mathbf{z}_2$ for some $\mathbf{z}_1$ and $\mathbf{z}_2$, whence $\mathsf{E}!|\mathbf{z}_1,\mathbf{z}_2|\wedge E|\mathbf{z}_1,\mathbf{z}_2|\wedge M|\mathbf{z}_1,\mathbf{z}_2|$ by lemma 5(i), theorems 23(i) and 23(ii), and the definition of S. Since $\mathbf{z}_3\in|\mathbf{z}_1,\mathbf{z}_2|\rightarrow\mathbf{z}_3\in\mathbf{x}$ by lemma 4 and the definition of $|\mathbf{z}_1,\mathbf{z}_2|$, it follows that $|\mathbf{z}_1,\mathbf{z}_2|\subseteq\mathbf{x}$ by the definition of $\subseteq$.

2 By the hypothesis and lemmas 3(i) and 5(i), $M\mathbf{x}\wedge\mathsf{E}!\mathbf{x}$, whence by lemma 6(ii) $\mathbf{x}\subseteq\mathbf{x}$. For a reductio suppose that $\mathbf{x}_1\subseteq\mathbf{x}$ for some $\mathbf{x}_1\neq\mathbf{x}$. Then for some $\mathbf{z}$, $\mathbf{z}\notin\mathbf{x}_1\wedge\mathbf{z}\in\mathbf{x}$ by the definition of $\subset$ and lemma 6(iii). Also $\mathbf{z}\in\mathbf{x}_1\rightarrow\mathbf{z}\in\mathbf{x}$ and $m\mathbf{z}_3\,\mathbf{z}_3\in\mathbf{x}_1$ by the definitions of $\subseteq$ and M. Since by the hypothesis $\mathbf{z}\in\mathbf{x}\leftrightarrow(\mathbf{z}=\mathbf{z}_1\vee\mathbf{z}=\mathbf{z}_2)$, it follows that $\mathbf{z}\in\mathbf{x}_1\rightarrow(\mathbf{z}=\mathbf{z}_1\vee\mathbf{z}=\mathbf{z}_2)$ whence $\mathbf{z}_1\notin\mathbf{x}_1\vee\mathbf{z}_2\notin\mathbf{x}_1$. Suppose $\mathbf{z}_1\notin\mathbf{x}_1$. Then $\mathbf{z}\in\mathbf{x}_1\rightarrow\mathbf{z}=\mathbf{z}_2$, whence $\neg m\mathbf{z}_3\,\mathbf{z}_3\in\mathbf{x}_1$. Contradiction. Suppose instead $\mathbf{z}_2\notin\mathbf{x}_1$. Then $\mathbf{z}\in\mathbf{x}_1\rightarrow\mathbf{z}=\mathbf{z}_1$, whence $\neg m\mathbf{z}_3\,\mathbf{z}_3\in\mathbf{x}_1$. Contradiction. So either way there is a contradiction, whence $\exists_1\mathbf{z}\,\mathbf{z}\subseteq\mathbf{x}$. Hence $\mathbf{x}=|\mathbf{z}_1,\mathbf{z}_2|$, whence $\exists_1\mathbf{z}(E\mathbf{z}\wedge\mathbf{z}\subseteq\mathbf{x})$. Hence $\iota\mathbf{z}(E\mathbf{z}\wedge\mathbf{z}\subseteq\mathbf{x})=\mathbf{x}$, whence $P(\mathbf{x})=\mathbf{x}$ by lemma 2(iii) and the definition of $P(\mathbf{x})$.

PROOF OF (ii). By the hypothesis $\mathbf{z}_1\in\mathbf{x}\wedge\mathbf{z}_2\in\mathbf{x}\wedge\mathbf{z}_3\in\mathbf{x}\wedge\mathbf{z}_1\neq\mathbf{z}_2\wedge\mathbf{z}_1\neq\mathbf{z}_3\wedge\mathbf{z}_2\neq\mathbf{z}_3$ for some $\mathbf{z}_1$, $\mathbf{z}_2$, $\mathbf{z}_3$, whence $E\mathbf{z}_1\wedge E\mathbf{z}_2\wedge E\mathbf{z}_3$ by lemma 5(i). Hence $\mathsf{E}!|\mathbf{z}_1,\mathbf{z}_2|\wedge\mathsf{E}!|\mathbf{z}_2,\mathbf{z}_3|$ by theorem 23(i), whence $M|\mathbf{z}_1,\mathbf{z}_2|\wedge M|\mathbf{z}_2,\mathbf{z}_3|\wedge\mathbf{z}\in|\mathbf{z}_1,\mathbf{z}_2|\leftrightarrow(\mathbf{z}=\mathbf{z}_1\vee\mathbf{z}=\mathbf{z}_2)\wedge\mathbf{z}\in|\mathbf{z}_2,\mathbf{z}_3|\leftrightarrow(\mathbf{z}=\mathbf{z}_2\vee\mathbf{z}=\mathbf{z}_3)$ by lemma 4 and the definitions of $|\mathbf{z}_1,\mathbf{z}_2|$ and $|\mathbf{z}_2,\mathbf{z}_3|$. Hence $\mathbf{z}_1\in|\mathbf{z}_1,\mathbf{z}_2|\wedge\mathbf{z}_1\notin|\mathbf{z}_2,\mathbf{z}_3|\wedge\mathbf{z}\in|\mathbf{z}_1,\mathbf{z}_2|\rightarrow\mathbf{z}\in\mathbf{x}\wedge\mathbf{z}\in|\mathbf{z}_2,\mathbf{z}_3|\rightarrow\mathbf{z}\in\mathbf{x}$, whence $|\mathbf{z}_1,\mathbf{z}_2|\neq|\mathbf{z}_2,\mathbf{z}_3|$ and also $|\mathbf{z}_1,\mathbf{z}_2|\subseteq\mathbf{x}\wedge|\mathbf{z}_2,\mathbf{z}_3|\subseteq\mathbf{x}$ by the definition of $\subseteq$. Since $E|\mathbf{z}_1,\mathbf{z}_2|\wedge E|\mathbf{z}_2,\mathbf{z}_3|$ by theorem 23(ii) and the definition of S, it follows that $m\mathbf{z}(E\mathbf{z}\wedge\mathbf{z}\subseteq\mathbf{x})$. Hence $P(\mathbf{x})=\mathbf{y}\cdot\mathbf{y}\subseteq\mathbf{x}$ by lemma 2(ii) and the definition of $P(\mathbf{x})$.

PROOF OF (iii).

1 For the $\rightarrow$ half, suppose $M\mathbf{x}$. Then by the definition of M, either $\exists\mathbf{y}_1\exists\mathbf{y}_2(\mathbf{y}_1\in\mathbf{x}\wedge\mathbf{y}_2\in\mathbf{x}\wedge\mathbf{y}_1\neq\mathbf{y}_2\wedge\forall\mathbf{y}_3(\mathbf{y}_3\in\mathbf{x}\rightarrow(\mathbf{y}_3=\mathbf{y}_1\vee\mathbf{y}_3=\mathbf{y}_2))$ or $\exists\mathbf{y}_1\exists\mathbf{y}_2\exists\mathbf{y}_3(\mathbf{y}_1\in\mathbf{x}\wedge\mathbf{y}_2\in\mathbf{x}\wedge\mathbf{y}_3\in\mathbf{x}\wedge\mathbf{y}_1\neq\mathbf{y}_2\wedge\mathbf{y}_1\neq\mathbf{y}_3\wedge\mathbf{y}_2\neq\mathbf{y}_3)$. Suppose $\exists\mathbf{y}_1\exists\mathbf{y}_2(\mathbf{y}_1\in\mathbf{x}\wedge\mathbf{y}_2\in\mathbf{x}\wedge\mathbf{y}_1\neq\mathbf{y}_2\wedge\forall\mathbf{y}_3(\mathbf{y}_3\in\mathbf{x}\rightarrow(\mathbf{y}_3=\mathbf{y}_1\vee\mathbf{y}_3=\mathbf{y}_2)))$. Then $P(\mathbf{x})=\mathbf{x}$ by theorem 25(i), whence $M(P(\mathbf{x}))$. Suppose instead $\exists\mathbf{y}_1\exists\mathbf{y}_2\exists\mathbf{y}_3(\mathbf{y}_1\in\mathbf{x}\wedge\mathbf{y}_2\in\mathbf{x}\wedge\mathbf{y}_3\in\mathbf{x}\wedge\mathbf{y}_1\neq\mathbf{y}_2\wedge\mathbf{y}_1\neq\mathbf{y}_3\wedge\mathbf{y}_2\neq\mathbf{y}_3)$. Then $P(\mathbf{x})=\mathbf{y}\cdot\mathbf{y}\subseteq\mathbf{x}$ by theorem 25(ii), whence $M(P(\mathbf{x}))$ by lemma 4.

2 For the $\leftarrow$ half, suppose $M(P(\mathbf{x}))$. Then $\mathsf{E}!(\mathrm{y}{:}\mathrm{y}\subseteq\mathrm{x})$ by lemma 3(i) and the definition of $P(\mathbf{x})$, whence $\exists\mathrm{z}\,\mathbf{z}\subseteq\mathbf{x}$ by lemma 2(iv). Hence $M\mathbf{x}$ by lemma 6(i).

PROOF OF (iv).

1 For the $\rightarrow$ half, suppose $S\mathbf{x}$. Then by the definitions of S and M, either $\exists\mathbf{y}_1\exists\mathbf{y}_2(\mathbf{y}_1\in\mathbf{x}\wedge\mathbf{y}_2\in\mathbf{x}\wedge\mathbf{y}_1\neq\mathbf{y}_2\wedge\forall\mathbf{y}_3(\mathbf{y}_3\in\mathbf{x}\rightarrow(\mathbf{y}_3=\mathbf{y}_1\vee\mathbf{y}_3=\mathbf{y}_2)))$ or $\exists\mathbf{y}_1\exists\mathbf{y}_2\exists\mathbf{y}_3(\mathbf{y}_1\in\mathbf{x}\wedge\mathbf{y}_2\in\mathbf{x}\wedge\mathbf{y}_3\in\mathbf{x}\wedge\mathbf{y}_1\neq\mathbf{y}_2\wedge\mathbf{y}_1\neq\mathbf{y}_3\wedge\mathbf{y}_2\neq\mathbf{y}_3)$. Suppose $\exists\mathbf{y}_1\exists\mathbf{y}_2(\mathbf{y}_1\in\mathbf{x}\wedge\mathbf{y}_2\in\mathbf{x}\wedge\mathbf{y}_1\neq\mathbf{y}_2\wedge\forall\mathbf{y}_3(\mathbf{y}_3\in\mathbf{x}\rightarrow(\mathbf{y}_3=\mathbf{y}_1\vee\mathbf{y}_3=\mathbf{y}_2)))$. Then $P(\mathbf{x})=\mathbf{x}$ by theorem 25(i), whence $S(P(\mathbf{x}))$. Suppose instead $\exists\mathbf{y}_1\exists\mathbf{y}_2\exists\mathbf{y}_3(\mathbf{y}_1\in\mathbf{x}\wedge\mathbf{y}_2\in\mathbf{x}\wedge\mathbf{y}_3\in\mathbf{x}\wedge\mathbf{y}_1\neq\mathbf{y}_2\wedge\mathbf{y}_1\neq\mathbf{y}_3\wedge\mathbf{y}_2\neq\mathbf{y}_3)$. Then $m\mathbf{z}(E\mathbf{z}\wedge\mathbf{z}\subseteq\mathbf{x})\wedge P(\mathbf{x})=\mathbf{y}\cdot\mathbf{y}\subseteq\mathbf{x}$ by theorem 25(ii). Since $S\mathbf{x}$, it follows by theorem 22(i) that $\mathbf{x}\in\mathbf{v}$ for some level $\mathbf{v}$, whence $\mathbf{y}\subseteq\mathbf{x}\rightarrow\mathbf{y}\in\mathbf{v}$ by theorem 9. Hence $\exists\mathbf{u}\forall\mathbf{y}((E\mathbf{y}\wedge\mathbf{y}\subseteq\mathbf{x})\rightarrow\mathbf{y}\in\mathbf{u})$, whence $S(P(\mathbf{x}))$ by theorem 1(iii).

2 For the $\leftarrow$ half, suppose $S(P(\mathbf{x}))$. Then $M(P(\mathbf{x}))$ by the definition of S, whence $M\mathbf{x}$ by theorem 25(iii). By the definition of M, either $\exists\mathbf{y}_1\exists\mathbf{y}_2(\mathbf{y}_1\in\mathbf{x}\wedge\mathbf{y}_2\in\mathbf{x}\wedge\mathbf{y}_1\neq\mathbf{y}_2\wedge\forall\mathbf{y}_3(\mathbf{y}_3\in\mathbf{x}\rightarrow(\mathbf{y}_3=\mathbf{y}_1\vee\mathbf{y}_3=\mathbf{y}_2)))$ or $\exists\mathbf{y}_1\exists\mathbf{y}_2\exists\mathbf{y}_3(\mathbf{y}_1\in\mathbf{x}\wedge\mathbf{y}_2\in\mathbf{x}\wedge\mathbf{y}_3\in\mathbf{x}\wedge\mathbf{y}_1\neq\mathbf{y}_2\wedge\mathbf{y}_1\neq\mathbf{y}_3\wedge\mathbf{y}_2\neq\mathbf{y}_3)$. Suppose $\exists\mathbf{y}_1\exists\mathbf{y}_2(\mathbf{y}_1\in\mathbf{x}\wedge\mathbf{y}_2\in\mathbf{x}\wedge\mathbf{y}_1\neq\mathbf{y}_2\wedge\forall\mathbf{y}_3(\mathbf{y}_3\in\mathbf{x}\rightarrow(\mathbf{y}_3=\mathbf{y}_1\vee\mathbf{y}_3=\mathbf{y}_2)))$. Then $P(\mathbf{x})=\mathbf{x}$ by theorem 25(i), whence $S\mathbf{x}$.

3 Suppose instead $\exists\mathbf{y}_1\exists\mathbf{y}_2\exists\mathbf{y}_3(\mathbf{y}_1\in\mathbf{x} \wedge \mathbf{y}_2\in\mathbf{x} \wedge \mathbf{y}_3\in\mathbf{x} \wedge \mathbf{y}_1\neq\mathbf{y}_2 \wedge \mathbf{y}_1\neq\mathbf{y}_3 \wedge \mathbf{y}_2\neq\mathbf{y}_3)$. Then $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\subseteq\mathbf{x}) \wedge P(\mathbf{x})=\mathbf{y}\cdot\mathbf{y}\subseteq\mathbf{x}$ by theorem 25(ii). Hence by lemma 4 $\mathbf{y}\in P(\mathbf{x})\leftrightarrow(E\mathbf{y} \wedge \mathbf{y}\subseteq\mathbf{x})$, whence $\mathbf{y}\cdot\mathbf{y}\in P(\mathbf{x}) \equiv \mathbf{y}\cdot(E\mathbf{y} \wedge \mathbf{y}\subseteq\mathbf{x})$ by lemma 1(i). Hence $P(\mathbf{x}) = \mathbf{y}\cdot(E\mathbf{y} \wedge \mathbf{y}\subseteq\mathbf{x})$ by lemma 5(ii), whence $S(\mathbf{y}{:}E\mathbf{y} \wedge \mathbf{y}\subseteq\mathbf{x})$ by lemma 2(ii).

4 By the definitions of $\subseteq$ and S, $\forall\mathbf{y}((E\mathbf{y} \wedge \mathbf{y}\subseteq\mathbf{x})\rightarrow S\mathbf{y})$, whence $S(\cup\mathbf{y}(E\mathbf{y} \wedge \mathbf{y}\subseteq\mathbf{x}))$ and $M(\cup\mathbf{y}(E\mathbf{y} \wedge \mathbf{y}\subseteq\mathbf{x}))$ by theorem 21(ii) and the definition of S. Hence $\cup\mathbf{y}(E\mathbf{y} \wedge \mathbf{y}\subseteq\mathbf{x}) = \cup\mathbf{y}(E\mathbf{y} \wedge \mathbf{y}\subseteq\mathbf{x}))$ by lemma 3(i), whence $\mathbf{z}_1\in\cup\mathbf{y}(E\mathbf{y} \wedge \mathbf{y}\subseteq\mathbf{x}) \leftrightarrow \exists\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\subseteq\mathbf{x} \wedge \mathbf{z}_1\in\mathbf{z})$ by the definition of $\cup\mathbf{y}(E\mathbf{y} \wedge \mathbf{y}\subseteq\mathbf{x})$, lemmas 4 and 5(i), and the definition of $\subseteq$.

5 We next prove that $\exists\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\subseteq\mathbf{x} \wedge \mathbf{z}_1\in\mathbf{z}) \leftrightarrow \mathbf{z}_1\in\mathbf{x}$. The $\rightarrow$ half follows immediately from the definition of $\subseteq$. For the $\leftarrow$ half, suppose $\mathbf{z}_1\in\mathbf{x}$. Then $\mathbf{z}_2\in\mathbf{x}$ for some $\mathbf{z}_2\neq\mathbf{z}_1$ by the definition of M, and also $E\mathbf{z}_1 \wedge E\mathbf{z}_2$ by lemma 5(i). Hence $\mathsf{E}!|\mathbf{z}_1, \mathbf{z}_2| \wedge E|\mathbf{z}_1, \mathbf{z}_2| \wedge M|\mathbf{z}_1, \mathbf{z}_2|$ by theorems 23(i) and 23(ii), and the definition of S. By lemma 4 and the definition of $|\mathbf{z}_1, \mathbf{z}_2|$ it follows that $\mathbf{z}_1\in|\mathbf{z}_1, \mathbf{z}_2|$ and $\mathbf{z}_3\in|\mathbf{z}_1, \mathbf{z}_2|\rightarrow\mathbf{z}_3\in\mathbf{x}$, whence $\exists\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\subseteq\mathbf{x} \wedge \mathbf{z}_1\in\mathbf{z})$ by the definition of $\subseteq$.

6 Since $\mathbf{z}_1\in\cup\mathbf{y}(E\mathbf{y} \wedge \mathbf{y}\subseteq\mathbf{x}) \leftrightarrow \exists\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\subseteq\mathbf{x} \wedge \mathbf{z}_1\in\mathbf{z})$ and $\exists\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\subseteq\mathbf{x} \wedge \mathbf{z}_1\in\mathbf{z}) \leftrightarrow \mathbf{z}_1\in\mathbf{x}$, it follows that $\mathbf{z}_1\in\cup\mathbf{y}(E\mathbf{y} \wedge \mathbf{y}\subseteq\mathbf{x}) \leftrightarrow \mathbf{z}_1\in\mathbf{x}$. Hence $\cup\mathbf{y}(E\mathbf{y} \wedge \mathbf{y}\subseteq\mathbf{x}) = \mathbf{x}$ by axiom 1(ii), whence $S\mathbf{x}$.

THEOREM 26. *Power-plus multitude*

(i) $M\mathbf{x} \leftrightarrow M(P^+(\mathbf{x}))$

(ii) $S\mathbf{x} \leftrightarrow S(P^+(\mathbf{x}))$

PROOF OF (i).

1 For the $\rightarrow$ half, suppose $M\mathbf{x}$. Then $m\mathbf{y}(E\mathbf{y} \wedge \mathbf{y}\in\mathbf{x})$ by the definition of M and lemma 5(i); a fortiori $m\mathbf{y}(E\mathbf{y} \wedge (\mathbf{y}\in\mathbf{x} \vee \mathbf{y}\subseteq\mathbf{x}))$. Hence $M(\mathbf{y}\cdot\mathbf{y}\in\mathbf{x} \vee \mathbf{y}\subseteq\mathbf{x})$ by lemma 2(i) and the definition of $\mathbf{y}\cdot(\mathbf{y}\in\mathbf{x} \vee \mathbf{y}\subseteq\mathbf{x})$, whence $M(P^+(\mathbf{x}))$ by the definition of $P^+(\mathbf{x})$.

2 For the $\leftarrow$ half, suppose $M(P^+(\mathbf{x}))$. Then $m\mathbf{y}(\mathbf{y}\in\mathbf{x} \vee \mathbf{y}\subseteq\mathbf{x}))$ by lemmas 2(i) and 3(i) and the definition of $P^+(\mathbf{x})$, whence $\exists\mathbf{y}\ \mathbf{y}\in\mathbf{x} \vee \exists\mathbf{y}\ \mathbf{y}\subseteq\mathbf{x}$. Suppose $\exists\mathbf{y}\ \mathbf{y}\in\mathbf{x}$. Then $M\mathbf{x}$ by axiom 1(iv). Suppose $\exists\mathbf{y}\ \mathbf{y}\subseteq\mathbf{x}$. Then $M\mathbf{x}$ by lemma 6(i).

PROOF OF (ii).

1 For the $\rightarrow$ half, suppose $S\mathbf{x}$. Then $M\mathbf{x}$ by the definition of S, whence $M(P^+(\mathbf{x}))$ by theorem 26(i). Hence $m\mathbf{y}(E\mathbf{y} \wedge (\mathbf{y}\in\mathbf{x} \vee \mathbf{y}\subseteq\mathbf{x}))$ by lemmas 2(i) and 3(i) and the definition of $P^+(\mathbf{x})$. Also from $M\mathbf{x}$ it follows that $\mathbf{x}\in\mathbf{v}$ for some level $\mathbf{v}$ by theorem 22(i). So $\mathbf{y}\in\mathbf{x}\rightarrow\mathbf{y}\in\mathbf{v}$ by theorem 8, and $\mathbf{y}\subseteq\mathbf{x}\rightarrow\mathbf{y}\in\mathbf{v}$ by theorem 9. Hence $\exists\mathbf{u}(\forall\mathbf{y}((E\mathbf{y} \wedge (\mathbf{y}\in\mathbf{x} \vee \mathbf{y}\subseteq\mathbf{x}))\rightarrow\mathbf{y}\in\mathbf{u}))$. From $m\mathbf{y}(E\mathbf{y} \wedge (\mathbf{y}\in\mathbf{x} \vee \mathbf{y}\subseteq\mathbf{x}))$ and $\exists\mathbf{u}(\forall\mathbf{y}((E\mathbf{y} \wedge (\mathbf{y}\in\mathbf{x} \vee \mathbf{y}\subseteq\mathbf{x}))\rightarrow\mathbf{y}\in\mathbf{u}))$ it follows that $S(\mathbf{y}\cdot\mathbf{y}\in\mathbf{x} \vee \mathbf{y}\subseteq\mathbf{x})$ by theorem 1(iii), whence $S(P^+(\mathbf{x}))$ by the definition of $P^+(\mathbf{x})$.

2 For the $\leftarrow$ half, suppose $S(P^+(\mathbf{x}))$. Then $M(P^+(\mathbf{x}))$ by the definition of S, whence $M\mathbf{x}$ by theorem 26(i). Also from $M(P^+(\mathbf{x}))$ it follows that $\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow\mathbf{y}\in P^+(\mathbf{x}))$ by lemmas 3(i), 4 and 5(i), and the definition of $P^+(\mathbf{x})$. Hence $\mathbf{x}\subseteq P^+(\mathbf{x})$ by the definition of $\subseteq$, whence $S\mathbf{x}$ by axiom 2(iii).

THEOREM 27. *Membership and proper submultitude among levels* $\mathbf{v}\in\mathbf{w}\leftrightarrow\mathbf{v}\subset\mathbf{w}$

PROOF.

1 For the $\rightarrow$ half, suppose $\mathbf{v}\in\mathbf{w}$. By lemma 8(i) $M\mathbf{v}$, whence $\mathbf{v}\subseteq\mathbf{w}$ by the corollary of theorem 8. By axiom 1(iii) $\mathbf{v}\neq\mathbf{w}$. So $\mathbf{v}\subset\mathbf{w}$ by the definition of $\subset$.

2 For the $\leftarrow$ half, suppose $\mathbf{v}\subset\mathbf{w}$, then $\mathbf{v}\subseteq\mathbf{w}$ and $\mathbf{v}\neq\mathbf{w}$ by the definition of $\subset$, whence by theorem 14 $\mathbf{v}\in\mathbf{w} \vee \mathbf{w}\in\mathbf{v}$. For a reductio suppose $\mathbf{w}\in\mathbf{v}$. Since $\mathbf{v}\subseteq\mathbf{w}$, it follows by the definition of $\subseteq$ that $\mathbf{w}\in\mathbf{w}$, contrary to axiom 1(iii). Hence $\mathbf{w}\notin\mathbf{v}$, whence $\mathbf{v}\in\mathbf{w}$.

THEOREM 28. *Numbers of individuals*

(i) $m\mathbf{x}U\mathbf{x}\leftrightarrow\exists\mathbf{x}M\mathbf{x}$

(ii) $m\mathbf{x}U\mathbf{x}\leftrightarrow\exists\mathbf{x}V\mathbf{x}$

(iii) $m\mathbf{x}U\mathbf{x}\leftrightarrow\exists\mathbf{x}S\mathbf{x}$

(iv) $m\mathbf{x}U\mathbf{x}\leftrightarrow m\mathbf{x}\ \mathbf{x}=\mathbf{x}$

(v) $\exists_1\mathbf{x}U\mathbf{x}\leftrightarrow\exists_1\mathbf{x}\ \mathbf{x}=\mathbf{x}$

(vi) $\neg\exists\mathbf{x}U\mathbf{x}\leftrightarrow\neg\exists\mathbf{x}\ \mathbf{x}=\mathbf{x}$

PROOF OF (i). For the $\rightarrow$ half, suppose $m\mathbf{x}U\mathbf{x}$. Then $m\mathbf{x}(E\mathbf{x} \wedge U\mathbf{x})$ by axiom 2(i), whence $\mathsf{E}!(\mathbf{x}\cdot U\mathbf{x})$ by lemma 2(i). Hence $\exists\mathbf{x}M\mathbf{x}$ by the definition of $\mathbf{x}\cdot U\mathbf{x}$. The $\leftarrow$ half is axiom 2(ii).

PROOF OF (ii). For the $\rightarrow$ half, suppose $m\mathbf{x}U\mathbf{x}$. Then $\exists\mathbf{x}M\mathbf{x}$ by theorem 28(i), whence $\exists\mathbf{x}V\mathbf{x}$ by lemma 9(iii), the strength of identity and the definition of V. For the $\leftarrow$ half, suppose $\exists\mathbf{x}V\mathbf{x}$. Then $\exists\mathbf{x}M\mathbf{x}$ by lemma 8(i), whence $m\mathbf{x}U\mathbf{x}$ by theorem 28(i).

PROOF OF (iii). For the $\rightarrow$ half, suppose $m\mathbf{x}U\mathbf{x}$. Then $\exists\mathbf{x}V\mathbf{x}$ by theorem 28(ii), whence $\exists\mathbf{x}S\mathbf{x}$ by lemma 8(iii). For the $\leftarrow$ half, suppose $\exists\mathbf{x}S\mathbf{x}$. Then $\exists\mathbf{x}M\mathbf{x}$ by the definition of S, whence $m\mathbf{x}U\mathbf{x}$ by theorem 28(i).

PROOF OF (iv). The $\rightarrow$ half is immediate. For the $\leftarrow$ half, suppose $m\mathbf{x}\ \mathbf{x}=\mathbf{x}$. If $\neg\exists\mathbf{x}M\mathbf{x}$ then $\forall\mathbf{x}U\mathbf{x}$ by lemma 3(i), whence $m\mathbf{x}U\mathbf{x}$. If $\exists\mathbf{x}M\mathbf{x}$ then $m\mathbf{x}U\mathbf{x}$ by theorem 28(i).

PROOF OF (v). For the $\rightarrow$ half, suppose $\exists_1\mathbf{x}U\mathbf{x}$. Then $\neg m\mathbf{x}\ \mathbf{x}=\mathbf{x}$ by theorem 28(iv), whence $\exists_1\mathbf{x}\ \mathbf{x}=\mathbf{x}$. For the $\leftarrow$ half, suppose $\mathbf{y}=\mathbf{y}$ for some unique $\mathbf{y}$. By lemma 3(i), $U\mathbf{y} \vee M\mathbf{y}$. For a reductio suppose $M\mathbf{y}$. Then $m\mathbf{x}\ \mathbf{x}=\mathbf{x}$ by theorems 28(i) and 28(iv). Contradiction. Hence $U\mathbf{y}$, whence $\exists_1\mathbf{x}U\mathbf{x}$.

PROOF OF (vi). For the $\rightarrow$ half, suppose $\neg\exists\mathbf{x}U\mathbf{x}$. Then neither $m\mathbf{x}U\mathbf{x}$ nor $\exists_1\mathbf{x}U\mathbf{x}$, whence neither $m\mathbf{x}\ \mathbf{x}=\mathbf{x}$ nor $\exists_1\mathbf{x}\ \mathbf{x}=\mathbf{x}$ by theorems 28(iv) and 28(v). Hence $\neg\exists\mathbf{x}\ \mathbf{x}=\mathbf{x}$. The $\leftarrow$ half is immediate.

THEOREM 29. *The lowest level*

(i) $m\mathbf{x}U\mathbf{x}\leftrightarrow\mathsf{E}!V_1$

(ii) $\exists\mathbf{x}V\mathbf{x}\leftrightarrow\mathsf{E}!V_1$

(iii) $V_1\equiv\iota\mathbf{v}(\neg\exists\mathbf{w}\ \mathbf{w}\in\mathbf{v})$

PROOF OF (i).

1 For the $\rightarrow$ half, suppose $m\mathbf{z}U\mathbf{z}$. Then $V_1 = \mathbf{z}{\cdot}U\mathbf{z}$ by theorem 28(i) and lemma 9(iii), whence $\mathsf{E!}V_1$ by the strength of identity.

2 For the $\leftarrow$ half, suppose $\mathsf{E!}V_1$. Then $\exists\mathbf{x}M\mathbf{x}$ by lemma 9(ii), whence $m\mathbf{z}U\mathbf{z}$ by theorem 28(i).

PROOF OF (ii). Immediate from theorems 28(ii) and 29(i).

PROOF OF (iii). Suppose $\neg\,\mathsf{E!}V_1$. Then $\neg\exists\mathbf{x}V\mathbf{x}$ by theorem 29(ii), whence $\neg\mathsf{E!}\iota\mathbf{v}(\neg\exists\mathbf{w}\ \mathbf{w}{\in}\mathbf{v})$. Hence $V_1 \equiv \iota\mathbf{v}(\neg\exists\mathbf{w}\ \mathbf{w}{\in}\mathbf{v})$. Suppose instead $\mathsf{E!}V_1$. Then $V(V_1)$ by the definition of V, whence $\exists\mathbf{u}\ \mathbf{u}{=}\mathbf{u}$. Hence $\exists_1\mathbf{v}(\mathbf{v}{=}\mathbf{v} \wedge \neg\exists\mathbf{w}(\mathbf{w}{\in}\mathbf{v} \wedge \mathbf{w}{=}\mathbf{w}))$ by theorem 15(ii). Hence $\mathsf{E!}\iota\mathbf{v}(\neg\exists\mathbf{w}\ \mathbf{w}{\in}\mathbf{v})$. By theorem 10(i), $\neg\exists\mathbf{w}\ \mathbf{w}{\in}V_1$. Hence $V_1 \equiv \iota\mathbf{v}\ (\neg\exists\mathbf{w}\ \mathbf{w}{\in}\mathbf{v})$.

THEOREM 30. *Levels next above I* $\quad \mathsf{E!}\mathbf{u} \leftrightarrow \mathsf{E!}\mathbf{u}'$

PROOF.

1 For the $\rightarrow$ half, suppose $\mathsf{E!}\mathbf{u}$. Then $\mathsf{E!}V^{\dagger}(\mathbf{u})$ by lemma 8(iii) and theorem 22(ii). Hence $\mathsf{E!}\iota\mathbf{x}(V\mathbf{u} \wedge \mathbf{x}{=}V^{\dagger}(\mathbf{u}))$, whence $\mathsf{E!}\mathbf{u}'$ by the definition of $\mathbf{u}'$.

2 For the $\leftarrow$ half , suppose $\mathsf{E!}\mathbf{u}'$, then $\mathbf{u}{\in}\mathbf{u}'$ by the definitions of $\mathbf{u}'$ and $V^{\dagger}(\mathbf{u})$. Hence $\mathsf{E!}\mathbf{u}$ by axiom 1(i).

THEOREM 31. *Levels next above II*

(i) $\mathbf{u}' = \mathbf{x}{\cdot}(U\mathbf{x} \vee \mathbf{x}{\subseteq}\mathbf{u})$

(ii) $\mathbf{u}' = P^{+}(\mathbf{u})$

PROOF OF (i). By lemma 8(i), $M\mathbf{u}$. Hence $\mathsf{E!}\mathbf{u}$ by lemma 3(i), whence $\mathsf{E!}\mathbf{u}'$ by theorem 30. By the definitions of $\mathbf{u}'$ and $V^{\dagger}(\mathbf{u})$, $\mathbf{u}{\in}\mathbf{u}'$ and $V\mathbf{u}'$. So $M\mathbf{u}'$ by lemma 8(i). We tackle three cases separately: (i) $\mathbf{u}'{=}V_1$, (ii) $\mathbf{u}'{=}V_2$ and (ii) $\mathbf{u}'{\neq}V_1$ and $\mathbf{u}'{\neq}V_2$.

Case (i) $\mathbf{u}'{=}V_1$

Since $\mathbf{u}{\in}\mathbf{u}'$, it follows that $\mathbf{u}{\in}V_1$. But $\mathbf{u}{\notin}V_1$ by theorem 10(i). Hence $\mathbf{u}' = \mathbf{x}{\cdot}(U\mathbf{x} \vee \mathbf{x}{\subseteq}\mathbf{u})$ by the tautology $A{\wedge}\neg A{\rightarrow}B$.

Case (ii) $\mathbf{u}'{=}V_2$

Since $\mathsf{E!}V_2$ it follows that $\iota\mathbf{w}(\mathbf{w}{\in}V_2) = V_1$ by theorem 10(ii). Since $\mathbf{u}{\in}V_2$, it follows that $\mathbf{u}{=}V_1$, whence $(U\mathbf{x} \vee \mathbf{x}{\subseteq}V_1){\leftrightarrow}(U\mathbf{x} \vee \mathbf{x}{\subseteq}\mathbf{u})$. By lemma 9(iv) $\mathbf{u}' = V_2 = \mathbf{x}{\cdot}(U\mathbf{x} \vee \mathbf{x}{\subseteq}V_1)$, whence by lemma 1(i) $\mathbf{u}' = \mathbf{x}{\cdot}(U\mathbf{x} \vee \mathbf{x}{\subseteq}\mathbf{u})$.

Case (iii) $\mathbf{u}'{\neq}V_1$ and $\mathbf{u}'{\neq}V_2$

1 We first prove $\mathbf{w}{\subseteq}\mathbf{u}{\leftrightarrow}\mathbf{w}{\in}\mathbf{u}'$. For the $\rightarrow$ half, suppose $\mathbf{w}{\subseteq}\mathbf{u}$. From $\mathbf{w}{\subseteq}\mathbf{u}$ and $\mathbf{u}{\in}\mathbf{u}'$ it follows by theorem 9 that $\mathbf{w}{\in}\mathbf{u}'$. For the $\leftarrow$ half, suppose $\mathbf{w}{\in}\mathbf{u}'$, then $\mathbf{u}{\notin}\mathbf{w}$ by the definitions of $\mathbf{u}'$ and $V^{\dagger}(\mathbf{u})$. Hence $\mathbf{w}{\in}\mathbf{u} \vee \mathbf{w}{=}\mathbf{u}$ by theorem 14. By lemma 8(i), $M\mathbf{w}$.

Suppose $\mathbf{w}{\in}\mathbf{u}$, then $\mathbf{w}{\subseteq}\mathbf{u}$ by the corollary of theorem 8. Suppose $\mathbf{w}{=}\mathbf{u}$, then $\mathbf{w}{\subseteq}\mathbf{u}$ by lemma 6(ii).

2 Since $\mathsf{E}!\mathbf{u}'$and $\mathbf{u}'{\neq}V_1$ and $\mathbf{u}'{\neq}V_2$, it follows by theorem 10(iii) that $\mathbf{w}{:}\mathbf{w}{\in}\mathbf{u}' = \mathbf{w}{\cdot}\mathbf{w}{\in}\mathbf{u}'$ and by theorem 11(ii) that $\mathbf{u}'{=}\mathrm{acc}(\mathbf{w}{:}\mathbf{w}{\in}\mathbf{u}')$, whence $\mathbf{u}'{=}\mathrm{acc}(\mathbf{w}{\cdot}\mathbf{w}{\in}\mathbf{u}')$. By the definition of acc, $\mathrm{acc}(\mathbf{w}{\cdot}\mathbf{w}{\in}\mathbf{u}') = \mathbf{x}{\cdot}(U\mathbf{x} \vee \exists\mathbf{y}(\mathbf{y}{\in}\mathbf{w}{\cdot}\mathbf{w}{\in}\mathbf{u}'{\wedge}\ (\mathbf{x}{\in}\mathbf{y} \vee \mathbf{x}{\subseteq}\mathbf{y})))$. Hence $\mathbf{x}{\in}\mathbf{u}'{\leftrightarrow}(E\mathbf{x} \wedge (U\mathbf{x} \vee \exists\mathbf{w_1}(\mathbf{w_1}{\in}\mathbf{u}' \wedge (\mathbf{x}{\in}\mathbf{w_1} \vee \mathbf{x}{\subseteq}\mathbf{w_1}))))$ by lemmas 4 and 5(i). We next prove $\mathbf{x}{\in}\mathbf{u}'{\leftrightarrow}(U\mathbf{x} \vee \mathbf{x}{\subseteq}\mathbf{u})$.

3 For the $\rightarrow$ half, suppose $\mathbf{x}{\in}\mathbf{u}'$, then $\mathsf{E}!\mathbf{x}$ by axiom 1(i), whence $U\mathbf{x} \vee M\mathbf{x}$ by lemma 3(i). Suppose $U\mathbf{x}$, a fortiori $U\mathbf{x} \vee \exists\mathbf{w_1}(\mathbf{w_1}{\in}\mathbf{u}' \wedge \mathbf{x}{\subseteq}\mathbf{w_1})$. Suppose instead that $M\mathbf{x}$. Since $\mathbf{x}{\in}\mathbf{u}'$, it follows that $U\mathbf{x} \vee \exists\mathbf{w_1}(\mathbf{w_1}{\in}\mathbf{u}' \wedge (\mathbf{x}{\in}\mathbf{w_1} \vee \mathbf{x}{\subseteq}\mathbf{w_1}))$, whence $U\mathbf{x} \vee \exists\mathbf{w_1}(\mathbf{w_1}{\in}\mathbf{u}' \wedge \mathbf{x}{\subseteq}\mathbf{w_1})$ by the corollary of theorem 8. Since $\mathbf{w}{\subseteq}\mathbf{u}{\leftrightarrow}\mathbf{w}{\in}\mathbf{u}'$, it follows that $U\mathbf{x} \vee \exists\mathbf{w_1}(\mathbf{w_1}{\subseteq}\mathbf{u} \wedge \mathbf{x}{\subseteq}\mathbf{w_1})$. So $U\mathbf{x} \vee \mathbf{x}{\subseteq}\mathbf{u}$ by the definition of $\subseteq$.

4 For the $\leftarrow$ half, suppose $U\mathbf{x}$. Since $M\mathbf{u}$ and $\mathsf{E}!\mathbf{u}$, it follows that $E\mathbf{x}$ by axioms 2(i) and 2(ii), whence $\mathbf{x}{\in}\mathbf{u}'$. Suppose instead that $\mathbf{x}{\subseteq}\mathbf{u}$, then $S\mathbf{x}$ by axiom 2(iii) and lemma 8(iii), whence $E\mathbf{x}$ by the definition of S. Since $\mathbf{u}{\subseteq}\mathbf{u}$ by lemma 6(ii), it follows that $E\mathbf{x} \wedge \exists\mathbf{w_1}(\mathbf{w_1}{\subseteq}\mathbf{u} \wedge \mathbf{x}{\subseteq}\mathbf{w_1})$. Since $\mathbf{w}{\subseteq}\mathbf{u}{\leftrightarrow}\mathbf{w}{\in}\mathbf{u}'$, it follows that $E\mathbf{x} \wedge \exists\mathbf{w_1}(\mathbf{w_1}{\in}\mathbf{u}' \wedge \mathbf{x}{\subseteq}\mathbf{w_1})$, whence $\mathbf{x}{\in}\mathbf{u}'$.

5 Since $\mathbf{x}{\in}\mathbf{u}'{\leftrightarrow}(U\mathbf{x} \vee \mathbf{x}{\subseteq}\mathbf{u})$, it follows by lemma 1(i) that $\mathbf{x}{\cdot}\mathbf{x}{\in}\mathbf{u}' \equiv \mathbf{x}{\cdot}(U\mathbf{x} \vee \mathbf{x}{\subseteq}\mathbf{u})$. Since $M\mathbf{u}'$, it follows by lemma 5(ii) that $\mathbf{u}' = \mathbf{x}{\cdot}(U\mathbf{x} \vee \mathbf{x}{\subseteq}\mathbf{u})$.

PROOF OF (ii).

1 By lemma 8(i), $M\mathbf{u}$. Hence by theorem 26(i) $M(P^{+}(\mathbf{u}))$, whence $\mathsf{E}!P^{+}(\mathbf{u})$ by lemma 3(i). So $P^{+}(\mathbf{u}) = \mathbf{x}{\cdot}(\mathbf{x}{\in}\mathbf{u} \vee \mathbf{x}{\subseteq}\mathbf{u})$ by the definition of $P^{+}(\mathbf{u})$, whence $\mathbf{x}{\in}P^{+}(\mathbf{u}){\leftrightarrow}(E\mathbf{x} \wedge (\mathbf{x}{\in}\mathbf{u} \vee \mathbf{x}{\subseteq}\mathbf{u}))$ by lemma 4. We shall prove $\mathbf{x}{\in}P^{+}(\mathbf{u}){\leftrightarrow}(U\mathbf{x} \vee \mathbf{x}{\subseteq}\mathbf{u})$.

2 For the $\rightarrow$ half, suppose $\mathbf{x}{\in}P^{+}(\mathbf{u})$, then $\mathbf{x}{\in}\mathbf{u} \vee \mathbf{x}{\subseteq}\mathbf{u}$. Suppose $\mathbf{x}{\in}\mathbf{u}$, then by axiom 1(i) $\mathsf{E}!\mathbf{x}$, whence by lemma 3(i) $U\mathbf{x} \vee M\mathbf{x}$. Suppose $U\mathbf{x}$; a fortiori $U\mathbf{x} \vee \mathbf{x}{\subseteq}\mathbf{u}$. Suppose $M\mathbf{x}$, then by the corollary of theorem 8 $\mathbf{x}{\subseteq}\mathbf{u}$; a fortiori $U\mathbf{x} \vee \mathbf{x}{\subseteq}\mathbf{u}$. Suppose instead that $\mathbf{x}{\subseteq}\mathbf{u}$, then again $U\mathbf{x} \vee \mathbf{x}{\subseteq}\mathbf{u}$.

3 For the $\leftarrow$ half, suppose $U\mathbf{x}$. Since $M\mathbf{u}$, it follows that $\mathsf{E}!\mathbf{u}$ by lemma 3(i). Hence $E\mathbf{x}$ by axioms 2(i) and 2(ii). Also by lemma 8(iv) $\mathbf{x}{\in}\mathbf{u}$, whence $\mathbf{x}{\in}P^{+}(\mathbf{u})$. Suppose instead that $\mathbf{x}{\subseteq}\mathbf{u}$, then $S\mathbf{x}$ by axiom 2(iii) and lemma 8(iii). Hence $E\mathbf{x}$ by the definition of S, whence $\mathbf{x}{\in}P^{+}(\mathbf{u})$.

4 Since $\mathbf{x}{\in}P^{+}(\mathbf{u}){\leftrightarrow}(U\mathbf{x} \vee \mathbf{x}{\subseteq}\mathbf{u})$, it follows by lemma 1(i) that $\mathbf{x}{\cdot}\mathbf{x}{\in}P^{+}(\mathbf{u}) \equiv \mathbf{x}{\cdot}(U\mathbf{x} \vee \mathbf{x}{\subseteq}\mathbf{u})$. Since $M(P^{+}(\mathbf{u}))$, it follows by lemma 5(ii) that $P^{+}(\mathbf{u}) = \mathbf{x}{\cdot}(U\mathbf{x} \vee \mathbf{x}{\subseteq}\mathbf{u})$, and so $\mathbf{u}' = P^{+}(\mathbf{u})$ by theorem 31(i).

THEOREM 32. *Sets and levels III* *Let* $S\mathbf{x}$, *then* $(V^{*}(\mathbf{x}))' = V^{\dagger}(\mathbf{x})$.

PROOF.

1 From the hypothesis it follows that $\mathsf{E}!V^{*}(\mathbf{x})$ by theorem 17(i), whence $V(V^{*}(\mathbf{x})) \wedge \mathbf{x}{\subseteq}V^{*}(\mathbf{x})$ by the definition of $V^{*}(\mathbf{x})$. Hence by theorem 30 $\mathsf{E}!(V^{*}(\mathbf{x}))'$, whence $V(V^{*}(\mathbf{x}))' \wedge V^{*}(\mathbf{x}){\in}(V^{*}(\mathbf{x}))'$ by the definitions of $(V^{*}(\mathbf{x}))'$ and $V^{\dagger}(V^{*}(\mathbf{x}))$. Hence $\mathbf{x}{\in}(V^{*}(\mathbf{x}))'$ by theorem 9.

2 For a reductio suppose that $\mathbf{x}\in\mathbf{w}$ for some $\mathbf{w}\in(V^*(\mathbf{x}))'$. By the definitions of $(V^*(\mathbf{x}))'$ and $V^\dagger(V^*(\mathbf{x}))$ it follows that $\neg\exists\mathbf{w_1}(\mathbf{w_1}\in(V^*(\mathbf{x}))' \wedge V^*(\mathbf{x})\in\mathbf{w_1})$. Hence $V^*(\mathbf{x})\notin\mathbf{w}$, whence by theorem 14 $V^*(\mathbf{x})=\mathbf{w} \vee \mathbf{w}\in V^*(\mathbf{x})$. Suppose $V^*(\mathbf{x})=\mathbf{w}$, then $\mathbf{x}\in V^*(\mathbf{x})$. Suppose instead that $\mathbf{w}\in V^*(\mathbf{x})$, then $\mathbf{x}\in V^*(\mathbf{x})$ by theorem 8. So either way $\mathbf{x}\in V^*(\mathbf{x})$, contrary to theorem 17(ii). Hence $\neg\exists\mathbf{w}(\mathbf{w}\in(V^*(\mathbf{x}))' \wedge \mathbf{x}\in\mathbf{w})$. By theorem 22(ii) $\mathsf{E}!V^\dagger(\mathbf{x})$. Since $V(V^*(\mathbf{x}))'$ and $\mathbf{x}\in(V^*(\mathbf{x}))'$ and $\neg\exists\mathbf{w}(\mathbf{w}\in(V^*(\mathbf{x}))' \wedge \mathbf{x}\in\mathbf{w})$, it follows that $(V^*(\mathbf{x}))' = V^\dagger(\mathbf{x})$ by the definition of $V^\dagger(\mathbf{x})$.

THEOREM 33. *Levels next above III*

(i) *Let* $\exists\mathbf{y_1}\exists\mathbf{y_2}(U\mathbf{y_1} \wedge U\mathbf{y_2} \wedge \mathbf{y_1}\neq\mathbf{y_2} \wedge \forall\mathbf{y_3}(U\mathbf{y_3}\rightarrow (\mathbf{y_3}=\mathbf{y_1} \vee \mathbf{y_3}=\mathbf{y_2})))$, *then* $\imath\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\subseteq V_1)=V_1 \wedge P(V_1)=V_1 \wedge \forall\mathbf{u}(\mathbf{u}\neq V_1\rightarrow(m\mathbf{z}\ \mathbf{z}\subseteq\mathbf{u} \wedge P(\mathbf{u})=\mathbf{y}\cdot\mathbf{y}\subseteq\mathbf{u}))$.

(ii) *Let* $\exists\mathbf{y_1}\exists\mathbf{y_2}\exists\mathbf{y_3}(U\mathbf{y_1} \wedge U\mathbf{y_2} \wedge U\mathbf{y_3} \wedge \mathbf{y_1}\neq\mathbf{y_2} \wedge \mathbf{y_1}\neq\mathbf{y_3} \wedge \mathbf{y_2}\neq\mathbf{y_3})$, *then* $\forall\mathbf{u}(m\mathbf{z}\ \mathbf{z}\subseteq\mathbf{u} \wedge P(\mathbf{u})=\mathbf{y}\cdot\mathbf{y}\subseteq\mathbf{u})$.

(iii) $\exists_1\mathbf{z}\ \mathbf{z}\subseteq\mathbf{u} \leftrightarrow \mathbf{u}'=V_1\oplus P(\mathbf{u})$

(iv) $m\mathbf{z}\ \mathbf{z}\subseteq\mathbf{u} \leftrightarrow \mathbf{u}'=V_1\cup P(\mathbf{u})$

PROOF OF (i).

1 By the hypothesis and theorem 29(i), $\mathsf{E}!V_1$. Hence $\exists\mathbf{x_1}\exists\mathbf{x_2}(\mathbf{x_1}\in V_1 \wedge \mathbf{x_2}\in V_1 \wedge \mathbf{x_1}\neq\mathbf{x_2} \wedge \forall\mathbf{x_3}(\mathbf{x_3}\in V_1\rightarrow (\mathbf{x_3}=\mathbf{x_1} \vee \mathbf{x_3}=\mathbf{x_2}))$ by the hypothesis and lemma 9(ii), whence $\imath\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\subseteq V_1)=V_1 \wedge P(V_1)=V_1$ by theorem 25(i).

2 From $\mathsf{E}!V_1$ it follows that $\exists\mathbf{u}\ \mathbf{u}\neq V_1$ by lemmas 9(vi) and 9(viii) and the definition of V. Consider an arbitrary level $\mathbf{v}\neq V_1$. Then by the definition of V and theorems 10(i) and 14, $V_1\in\mathbf{v}$. By the hypothesis and lemma 8(iv), $m\mathbf{z}(U\mathbf{z} \wedge \mathbf{z}\in\mathbf{v})$. Since $\neg U(V_1)$ by lemma 9(ii) and the definition of U, it follows that $\exists\mathbf{y_1}\exists\mathbf{y_2}\exists\mathbf{y_3}(\mathbf{y_1}\in\mathbf{v} \wedge \mathbf{y_2}\in\mathbf{v} \wedge \mathbf{y_3}\in\mathbf{v} \wedge \mathbf{y_1}\neq\mathbf{x_2} \wedge \mathbf{y_1}\neq\mathbf{y_3} \wedge \mathbf{y_2}\neq\mathbf{y_3})$, whence by theorem 25(ii) $m\mathbf{z}\ \mathbf{z}\subseteq\mathbf{v} \wedge P(\mathbf{v})=\mathbf{y}\cdot\mathbf{y}\subseteq\mathbf{v}$. Since $\mathbf{v}$ is arbitrary, we can generalize to get $\forall\mathbf{u}(\mathbf{u}\neq V_1\rightarrow(m\mathbf{z}\ \mathbf{z}\subseteq\mathbf{u} \wedge P(\mathbf{u})=\mathbf{y}\cdot\mathbf{y}\subseteq\mathbf{u}))$.

PROOF OF (ii). By the hypothesis and theorem 28(ii), $\exists\mathbf{x}V\mathbf{x}$. Consider an arbitrary level $\mathbf{v}$. Then by lemma 8(iv), $U\mathbf{y}\rightarrow\mathbf{y}\in\mathbf{v}$, whence by the hypothesis $\exists\mathbf{y_1}\exists\mathbf{y_2}\exists\mathbf{y_3}(\mathbf{y_1}\in\mathbf{v} \wedge \mathbf{y_2}\in\mathbf{v} \wedge \mathbf{y_3}\in\mathbf{v} \wedge \mathbf{y_1}\neq\mathbf{y_2} \wedge \mathbf{y_1}\neq\mathbf{y_3} \wedge \mathbf{y_2}\neq\mathbf{y_3})$. Hence by theorem 25(ii) $m\mathbf{z}\ \mathbf{z}\subseteq\mathbf{v} \wedge P(\mathbf{v})=\mathbf{y}\cdot\mathbf{y}\subseteq\mathbf{v}$. Since $\mathbf{v}$ is arbitrary, we can generalize to get $\forall\mathbf{u}(m\mathbf{z}\ \mathbf{z}\subseteq\mathbf{u} \wedge P(\mathbf{u})=\mathbf{y}\cdot\mathbf{y}\subseteq\mathbf{u})$.

PROOF OF (iii).

1 For the $\rightarrow$ half, suppose $\mathbf{z}\subseteq\mathbf{u}$ for some unique $\mathbf{z}$. Then $m\mathbf{x}U\mathbf{x}$ by the definition of $\subseteq$ and theorem 28(i), whence $\exists\mathbf{y_1}\exists\mathbf{y_2}(U\mathbf{y_1} \wedge U\mathbf{y_2} \wedge \mathbf{y_1}\neq\mathbf{y_2} \wedge \forall\mathbf{y_3}(U\mathbf{y_3}\rightarrow(\mathbf{y_3}=\mathbf{y_1} \vee \mathbf{y_3}=\mathbf{y_2}))$ by theorem 33(ii). Hence $\mathbf{u}=V_1 \wedge \imath\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\subseteq V_1)=V_1 \wedge P(V_1)=V_1$ by theorem 33(i). Hence $\mathsf{E}!V_1$ by the strength of identity, whence $V_1 = \mathbf{z}\cdot U\mathbf{z}$ by lemma 9(i). Hence MV_1 and $\mathbf{z}\in V_1\leftrightarrow(E\mathbf{z} \wedge U\mathbf{z})$ by lemma 4, whence $E(P(\mathbf{u}))$ by the definitions of V and S, lemma 8(iii) and theorem 25(iv).

2 By theorems 30 and 31(i), and lemma 4, $M\mathbf{u}'$ and $\mathbf{z}\in\mathbf{u}'\leftrightarrow(E\mathbf{z} \wedge (U\mathbf{z} \vee \mathbf{z}\subseteq\mathbf{u}))$, whence $\mathbf{z}\in\mathbf{u}'\leftrightarrow(MV_1 \wedge E(P(\mathbf{u})) \wedge (\mathbf{z}\in V_1 \vee \mathbf{z}=P(\mathbf{u})))$, and also $\mathbf{u}'=\mathbf{x}\cdot\mathbf{x}\in\mathbf{u}'$ by lemma 5(ii). Hence $\mathbf{u}'=\mathbf{x}\cdot(MV_1 \wedge E(P(\mathbf{u})) \wedge (\mathbf{z}\in V_1 \vee \mathbf{z}=P(\mathbf{u})))$ by lemma 1(i), whence $\mathbf{u}'=V_1\oplus P(\mathbf{u})$ by the definition of $V_1\oplus P(\mathbf{u})$.

3 For the $\leftarrow$ half, suppose $\mathbf{u}'=V_1\oplus P(\mathbf{u})$. Then $\mathsf{E}!\mathbf{u}$ by the strength of identity and theorem 30, whence $\mathbf{z}\in V_1\leftrightarrow U\mathbf{z}$ by theorems 28(ii) and 29(ii), and lemma 9(ii). Also

$\mathbf{x}\cdot(U\mathbf{x} \vee \mathbf{x}\subseteq\mathbf{u}) = \mathbf{x}\cdot(MV_1 \wedge E(P(\mathbf{u})) \wedge (\mathbf{x}\in V_1 \vee \mathbf{x}=P(\mathbf{u})))$ by theorem 31(i) and the definition of $V_1\oplus P(\mathbf{u})$. By axiom 2(iii), lemma 8(iii) and the definitions of S and U, it follows that $\mathbf{z}\subseteq\mathbf{u}\rightarrow E\mathbf{z} \wedge \neg U\mathbf{z}$. Hence by lemma 4 $\mathbf{z}\subseteq\mathbf{u}\rightarrow\mathbf{z}=P(\mathbf{u})$, whence $\neg m\mathbf{z}\ \mathbf{z}\subseteq\mathbf{u}$. Since $\mathbf{u}\subseteq\mathbf{u}$ by lemmas 6(ii) and 8(i), it follows that $\exists_1\mathbf{z}\ \mathbf{z}\subseteq\mathbf{u}$.

PROOF OF (iv).

1 For the $\rightarrow$ half, suppose $m\mathbf{z}\ \mathbf{z}\subseteq\mathbf{u}$. Then $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\subseteq\mathbf{u})$ by axiom 2(iii), lemma 8(iii) and the definition of S. Hence $P(\mathbf{u})=\mathbf{y}\cdot\mathbf{y}\subseteq\mathbf{u}$ by lemma 2(ii) and the definition of $P(\mathbf{u})$, whence $M(P(\mathbf{u}))$ and $\mathbf{z}\in P(\mathbf{u})\leftrightarrow(E\mathbf{z} \wedge \mathbf{z}\subseteq\mathbf{u})$ by lemma 4. Since $m\mathbf{z}\ \mathbf{z}\subseteq\mathbf{u}$, it follows by lemmas 3(i) and 6(i) that $\mathsf{E}!\mathbf{u} \wedge M\mathbf{u}$, whence $\exists\mathbf{x}M\mathbf{x}$. Hence by lemma 9(iii) $V_1 = \mathbf{z}\cdot U\mathbf{z}$, whence MV_1 and $\mathbf{z}\in V_1\leftrightarrow(E\mathbf{z} \wedge U\mathbf{z})$ by lemma 4. By theorems 30 and 31(i), and lemma 4, $M\mathbf{u}'$ and $\mathbf{z}\in\mathbf{u}'\leftrightarrow(E\mathbf{z} \wedge (U\mathbf{z} \vee \mathbf{z}\subseteq\mathbf{u}))$, whence $\mathbf{z}\in\mathbf{u}'\leftrightarrow(\mathbf{z}\in V_1 \vee \mathbf{z}\in P(\mathbf{u}))$.

2 Since MV_1 and $M(P(\mathbf{u}))$, it follows by theorem 20 that $M(V_1\cup P(\mathbf{u}))$, whence $\mathsf{E}!(V_1\cup P(\mathbf{u}))$ by lemma 3(i). By the definition of $\cup$, $V_1\cup P(\mathbf{u})) = \mathbf{z}\cdot(MV_1 \wedge M(P(\mathbf{u})) \wedge (\mathbf{z}\in V_1 \vee \mathbf{z}\in P(\mathbf{u})))$. Hence $\mathbf{z}\in(V_1\cup P(\mathbf{u}))\leftrightarrow(\mathbf{z}\in V_1 \vee \mathbf{z}\in P(\mathbf{u}))$ by lemmas 4 and 5(i). Since $\mathbf{z}\in\mathbf{u}'\leftrightarrow(\mathbf{z}\in V_1 \vee \mathbf{z}\in P(\mathbf{u}))$, it follows that $\mathbf{z}\in\mathbf{u}'\leftrightarrow\mathbf{z}\in(V_1\cup P(\mathbf{u}))$.

3 Since $M\mathbf{u}'$ and $M(V_1\cup P(\mathbf{u}))$ and $\mathbf{z}\in\mathbf{u}'\leftrightarrow\mathbf{z}\in(V_1\cup P(\mathbf{u}))$, it follows by axiom 1(ii) that $\mathbf{u}'=V_1\cup P(\mathbf{u})$.

4 For the $\leftarrow$ half, suppose $\mathbf{u}'=V_1\cup P(\mathbf{u})$. Then $\mathsf{E}!\mathbf{u}$ by the strength of identity and theorem 30, whence $m\mathbf{x}U\mathbf{x}$ and $\mathbf{z}\in V_1\leftrightarrow U\mathbf{z}$ by theorems 28(ii) and 29(ii), and lemma 9(ii). Also $\mathbf{x}\cdot(U\mathbf{x} \vee \mathbf{x}\subseteq\mathbf{u}) = \mathbf{x}\cdot(MV_1 \wedge M(P(\mathbf{u})) \wedge (\mathbf{x}\in V_1 \vee \mathbf{x}\in P(\mathbf{u})))$ by theorem 31(i) and the definition of $V_1\cup P(\mathbf{u})$. By axiom 2(iii), lemma 8(iii) and the definitions of S and U, it follows that $\mathbf{z}\subseteq\mathbf{u}\rightarrow E\mathbf{z} \wedge \neg U\mathbf{z}$. Hence by lemma 4 $\mathbf{z}\subseteq\mathbf{u}\rightarrow\mathbf{z}\in P(\mathbf{u})$. For a reductio suppose $\mathbf{z}_1\subseteq\mathbf{u}$ for some unique $\mathbf{z}_1$. Then by theorems 33(i) and 33(ii), $P(\mathbf{u})=V_1$, whence $U\mathbf{z}_1$. Contradiction. Since $\mathbf{u}\subseteq\mathbf{u}$ by lemmas 6(ii) and 8(i), it follows that $m\mathbf{z}\ \mathbf{z}\subseteq\mathbf{u}$.

THEOREM 34. *Limit levels*

(i) $m\mathbf{x}U\mathbf{x} \leftrightarrow \exists\mathbf{x}L\mathbf{x}$

(ii) $L\mathbf{u} \leftrightarrow \mathbf{u} = \cup\mathbf{v}\ \mathbf{v}\in\mathbf{u}$

PROOF OF (i). By axiom 2(vi) and the definition of L, $\exists\mathbf{x}V\mathbf{x} \leftrightarrow \exists\mathbf{x}L\mathbf{x}$. Hence by theorem 28(ii) $m\mathbf{x}U\mathbf{x} \leftrightarrow \exists\mathbf{x}L\mathbf{x}$.

PROOF OF (ii).

1 For the $\rightarrow$ half, suppose $L\mathbf{u}$. Then $M\mathbf{u}$ by lemma 8(i), whence $\mathsf{E}!\mathbf{u}$ by lemma 3(i). Also by the definition of L, $\mathbf{u}\neq V_1$. For a reductio suppose $\mathbf{u}=V_2$. Then $\mathsf{E}!V_2$ by the strength of identity, whence $V_2 = \mathbf{z}\cdot(U\mathbf{z} \vee \mathbf{z}\subseteq V_1)$ and $\mathsf{E}!V_1$ by lemmas 9(iv) and 9(vi). By theorem 31(i) and the definition of V, $V_1' = \mathbf{z}\cdot(U\mathbf{z} \vee \mathbf{z}\subseteq V_1)$. Hence $V_2=V_1'$, whence $\neg LV_2$ by the definition of L. Contradiction. Hence $\mathbf{u}\neq V_2$.

2 Since $\mathsf{E}!\mathbf{u}$ and $\mathbf{u}\neq V_1$ and $\mathbf{u}\neq V_2$, it follows by theorem 10(iii) that $m\mathbf{w}\ \mathbf{w}\in\mathbf{u}$. We shall prove $\mathbf{z}\in\mathbf{u}\leftrightarrow\exists\mathbf{w}(\mathbf{w}\in\mathbf{u} \wedge \mathbf{z}\in\mathbf{w})$. For the $\rightarrow$ half, suppose $\mathbf{z}\in\mathbf{u}$. By axiom 1(i) and lemma 3(i) $U\mathbf{z} \vee M\mathbf{z}$. Suppose $U\mathbf{z}$. Then $\forall\mathbf{v}\ \mathbf{z}\in\mathbf{v}$ by lemma 8(iv). Since $\exists\mathbf{w}\ \mathbf{w}\in\mathbf{u}$, it follows that $\exists\mathbf{w}(\mathbf{w}\in\mathbf{u} \wedge \mathbf{z}\in\mathbf{w})$. Suppose instead that $M\mathbf{z}$. Then $S\mathbf{z}$ by the definition of S

and lemma 5(i). By theorem 17(i) $\mathsf{E}!V^*(\mathbf{z})$, whence $V(V^*(\mathbf{z}))$ by the definition of $V^*(\mathbf{z})$. Hence $\mathsf{E}!(V^*(\mathbf{z}))'$ by theorem 30, whence $\mathbf{u}\neq(V^*(\mathbf{z}))'$ by the definition of L. By theorem 32 and the definition of $V^\dagger(\mathbf{z})$ it follows that $(V^*(\mathbf{z}))' = \iota\mathbf{v}(\mathbf{z}\in\mathbf{v} \wedge \neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{v} \wedge \mathbf{z}\in\mathbf{w}))$. Hence $\mathbf{u}\in(V^*(\mathbf{z}))' \vee (V^*(\mathbf{z}))'\in\mathbf{u}$ by theorem 14. For a reductio suppose $\mathbf{u}\in(V^*(\mathbf{z}))'$. Since $\mathbf{z}\in\mathbf{u}$, it follows that $\exists\mathbf{w}(\mathbf{w}\in(V^*(\mathbf{z}))' \wedge \mathbf{z}\in\mathbf{w})$. Contradiction. Hence $(V^*(\mathbf{z}))'\in\mathbf{u}$, whence $\exists\mathbf{w}(\mathbf{w}\in\mathbf{u} \wedge \mathbf{z}\in\mathbf{w})$. For the $\leftarrow$ half, suppose $\exists\mathbf{w}(\mathbf{w}\in\mathbf{u} \wedge \mathbf{z}\in\mathbf{w})$. Then $\mathbf{z}\in\mathbf{u}$ by theorem 8.

3 By lemma 8(i) $\forall\mathbf{w}(\mathbf{w}\in\mathbf{u}\rightarrow M\mathbf{w})$. Since $\exists\mathbf{w}\ \mathbf{w}\in\mathbf{u}$ it follows that $M(\cup\mathbf{v}\ \mathbf{v}\in\mathbf{u})$ by theorem 21(i), whence $\mathsf{E}!(\cup\mathbf{v}\ \mathbf{v}\in\mathbf{u})$ by lemma 3(i). Hence $\mathbf{z}\in(\cup\mathbf{v}\ \mathbf{v}\in\mathbf{u})\leftrightarrow\exists\mathbf{w}(\mathbf{w}\in\mathbf{u} \wedge \mathbf{z}\in\mathbf{w})$ by lemmas 4 and 5(i), and the definition of $\cup\mathbf{v}\ \mathbf{v}\in\mathbf{u}$. Since $\mathbf{z}\in\mathbf{u}\leftrightarrow\exists\mathbf{w}(\mathbf{w}\in\mathbf{u} \wedge \mathbf{z}\in\mathbf{w})$, it follows that $\mathbf{z}\in\mathbf{u}\leftrightarrow\mathbf{z}\in(\cup\mathbf{v}\ \mathbf{v}\in\mathbf{u})$, whence by axiom 1(ii) $\mathbf{u} = \cup\mathbf{v}\ \mathbf{v}\in\mathbf{u}$.

4 For the $\rightarrow$ half, suppose $\mathbf{u} = \cup\mathbf{v}\ \mathbf{v}\in\mathbf{u}$. Then $\mathsf{E}!(\cup\mathbf{v}\ \mathbf{v}\in\mathbf{u})$ by the strength of identity. For a reductio suppose $\mathbf{u}=V_1$. Then by theorem 10(i) $\neg\exists\mathbf{w}\ \mathbf{w}\in\mathbf{u}$, whence $\neg\mathsf{E}!(\cup\mathbf{v}\ \mathbf{v}\in\mathbf{u})$ by lemma 2(i) and the definition of $\cup\mathbf{v}\ \mathbf{v}\in\mathbf{u}$. Contradiction. Hence $\mathbf{u}\neq V_1$. For a reductio suppose $\mathbf{u}=\mathbf{y}'$ for some $\mathbf{y}$. Then $\mathbf{y}\in\mathbf{u} \wedge \neg\exists\mathbf{w}(\mathbf{w}\in\mathbf{u} \wedge \mathbf{y}\in\mathbf{w}))$ by the definitions of $\mathbf{y}'$ and $V^\dagger(\mathbf{y})$. But $\mathbf{z}\in(\cup\mathbf{v}\ \mathbf{v}\in\mathbf{u})\rightarrow\exists\mathbf{w}(\mathbf{w}\in\mathbf{u} \wedge \mathbf{z}\in\mathbf{w})$ by lemma 4 and the definition of $\cup\mathbf{v}\ \mathbf{v}\in\mathbf{u}$. Contradiction. Hence $\neg\exists\mathbf{y}\ \mathbf{u}=\mathbf{y}'$, whence $L\mathbf{u}$ by the definition of L.

THEOREM 35. *The lowest limit level* $\mathsf{E}!V_\omega \leftrightarrow \exists\mathbf{x}L\mathbf{x}$

PROOF. The $\rightarrow$ half follows immediately from the definition of V_ω. For the $\leftarrow$ half, suppose $\exists\mathbf{x}L\mathbf{x}$. Then $\exists_1\mathbf{x}(L\mathbf{x} \wedge \neg\exists\mathbf{y}(\mathbf{y}\in\mathbf{x} \wedge L\mathbf{y}))$ by the definition of L and theorem 15(ii), whence $\mathsf{E}!V_\omega$ by the definition of V_ω.

THEOREM 36. *V_ω is inductive* *Let* $\mathsf{E}!V_\omega$, *then* $I(V_\omega)$.

PROOF.

1 It follows from the hypothesis that $V(V_\omega)$ by the definitions of V_ω and L, whence MV_ω by lemma 8(i). Hence $V(V_1)$ by lemma 9(iii) and the definition of V. By theorem 14, $V_1\in V_\omega \vee V_1=V_\omega \vee V_\omega\in V_1$. But $V_1\neq V_\omega$ by the definitions of L and V_ω, and $V_\omega\notin V_1$ by theorem 10(i). Hence $V_1\in V_\omega$.

2 For a reductio suppose $\mathbf{u}\in V_\omega \wedge \mathbf{u}'\notin V_\omega$ for some $\mathbf{u}$. Then $\mathsf{E}!\mathbf{u}'$ by theorem 30, whence $V\mathbf{u}'$ and $V_\omega\notin\mathbf{u}'$ by the definitions of $\mathbf{u}'$ and $V^\dagger(\mathbf{u})$. Hence $\mathbf{u}'=V_\omega$ by theorem 14. But since $\exists\mathbf{x}\ \mathbf{u}'=\mathbf{x}'$ it follows that $\mathbf{u}'\neq V_\omega$ by the definitions of L and V_ω. Contradiction. Hence $\forall\mathbf{x}((V\mathbf{x} \wedge \mathbf{x}\in V_\omega)\rightarrow\mathbf{x}'\in V_\omega)$, whence $I(V_\omega)$ by the definition of I.

THEOREM 37. $\mathbf{N}^*$ *Let* $m\mathbf{x}U\mathbf{x}$, *then* $S(\mathbf{N}^*) \wedge I(\mathbf{N}^*)$.

PROOF.

1 From the hypothesis it follows that $\mathsf{E}!V_\omega$ by theorems 34(i) and 35, whence $S(V_\omega)$ by lemma 8(iii) and the definitions of V_ω and L, and also $I(V_\omega)$ by theorem 36. By the definitions of I and V it follows that $I\mathbf{y}\rightarrow(V_1\in\mathbf{y} \wedge V_1'\in\mathbf{y})$, whence $V_1\in V_\omega \wedge V_1'\in V_\omega$.

Hence $\mathsf{E}!V_1$ and $\mathsf{E}!V_1'$ by axiom 1(i). By the definitions of V_1' and $V^\dagger(V_1)$ it follows that $V_1 \in V_1'$. Hence $V_1 \neq V_1'$ by axiom 1(iii), whence $m\mathbf{x}(\mathbf{x} \in V_\omega \wedge \forall \mathbf{y}(I\mathbf{y} \rightarrow \mathbf{x} \in \mathbf{y}))$.

3 Since $S(V_\omega)$ and $m\mathbf{x}(\mathbf{x} \in V_\omega \wedge \forall \mathbf{y}(I\mathbf{y} \rightarrow \mathbf{x} \in \mathbf{y}))$, it follows that $S(\mathbf{x} \cdot\ \mathbf{x} \in V_\omega \wedge \forall \mathbf{y}(I\mathbf{y} \rightarrow \mathbf{x} \in \mathbf{y}))$ by theorem 1(i). Since $I(V_\omega)$ it follows that $(\mathbf{x} \in V_\omega \wedge \forall \mathbf{y}(I\mathbf{y} \rightarrow \mathbf{x} \in \mathbf{y})) \leftrightarrow (\forall \mathbf{y}(I\mathbf{y} \rightarrow \mathbf{x} \in \mathbf{y}) \wedge \exists \mathbf{z} I\mathbf{z})$, whence $S(\mathbf{x} \cdot\ \forall \mathbf{y}(I\mathbf{y} \rightarrow \mathbf{x} \in \mathbf{y}) \wedge \exists \mathbf{z} I\mathbf{z})$ by lemma 1(i). Hence by lemmas 2(i), 3(i) and 3(ii), $m\mathbf{x}(E\mathbf{x} \wedge \forall \mathbf{y}(I\mathbf{y} \rightarrow \mathbf{x} \in \mathbf{y}) \wedge \exists \mathbf{z} I\mathbf{z})$, Hence by lemma 2(ii) $\mathbf{x} \cdot (\forall \mathbf{y}(I\mathbf{y} \rightarrow \mathbf{x} \in \mathbf{y}) \wedge \exists \mathbf{z} I\mathbf{z}) = \mathbf{x} : (\forall \mathbf{y}(I\mathbf{y} \rightarrow \mathbf{x} \in \mathbf{y}) \wedge \exists \mathbf{z} I\mathbf{z})$, whence $S(\mathbf{x} :\ \forall \mathbf{y}(I\mathbf{y} \rightarrow \mathbf{x} \in \mathbf{y}) \wedge \exists \mathbf{z} I\mathbf{z})$. Hence $S(\cap \mathbf{x} I\mathbf{x})$ by the definition of $\cap \mathbf{x} I\mathbf{x}$, whence $S(\mathbf{N}^*)$ by the definition of $\mathbf{N}^*$.

4 By lemmas 3(i) and 3(ii) $E!\mathbf{N}^*$, whence $\mathbf{N}^* = \mathbf{x} \cdot (\forall \mathbf{y}(I\mathbf{y} \rightarrow \mathbf{x} \in \mathbf{y}) \wedge \exists \mathbf{z} I\mathbf{z})$. By the definition of V and lemma 8(ii), EV_1, whence $V_1 \in \mathbf{N}^*$ by lemma 4.

5 For a reductio suppose $\mathbf{u} \in \mathbf{N}^* \wedge \mathbf{u}' \notin \mathbf{N}^*$ for some level $\mathbf{u}$. Then $\forall \mathbf{y}(I\mathbf{y} \rightarrow \mathbf{u} \in \mathbf{y})$ by lemma 4. By the definition of I it follows that $\forall \mathbf{y}(I\mathbf{y} \rightarrow \forall \mathbf{v}(\mathbf{v} \in \mathbf{y} \rightarrow \mathbf{v}' \in \mathbf{y}))$, whence $\forall \mathbf{y}(I\mathbf{y} \rightarrow \mathbf{u}' \in \mathbf{y})$. By lemma 8(ii), theorem 30, and the definitions of $\mathbf{u}'$ and $V^\dagger(\mathbf{u})$, it follows that $E\mathbf{u}'$, whence $\mathbf{u}' \in \mathbf{N}^*$ by lemma 4. Contradiction. Hence $\forall \mathbf{v}(\mathbf{v} \in \mathbf{N}^* \rightarrow \mathbf{v}' \in \mathbf{N}^*)$, whence $I(\mathbf{N}^*)$ by the definition of I.

THEOREM 38. *Ordered pairs*
Let $m\mathbf{z}U\mathbf{z} \wedge \neg C\mathbf{x}_1 \wedge \neg C\mathbf{x}_2 \wedge \neg C\mathbf{y}_1 \wedge \neg C\mathbf{y}_2$. *Then* (i) $E!\langle \mathbf{x}_1, \mathbf{x}_2 \rangle$ *and* (ii) $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle = \langle \mathbf{y}_1, \mathbf{y}_2 \rangle \leftrightarrow (\mathbf{x}_1 \equiv \mathbf{y}_1 \wedge \mathbf{x}_2 \equiv \mathbf{y}_2)$.

We supply sketches for the interested reader to develop into full-dress proofs. The proof of (i) is by repeated application of pairing (theorem 23(i)), having established on each occasion that the members of the next putative pair are distinct elements. At the start it is shown that $E[\mathbf{x}_1, V_1]$ and $E[\mathbf{x}_1, V_2]$ and $[\mathbf{x}_1, V_1] \neq [\mathbf{x}_1, V_2]$, and similarly for $\mathbf{x}_2$. Four cases for each of $\mathbf{x}_1$ and $\mathbf{x}_2$ need to be tackled here, which between them exhaust the possibilities: zilch, V_1, V_2, any other element.

By definition an ordered pair $\langle a, b \rangle$ is of the form $|a^*, b^*|$, where a^* codes coordinate *a,* and b^* codes b. The markers V_1 and V_2 serve to distinguish the two. The proof of the $\rightarrow$ half of (ii) proceeds by showing that different coordinates have different codes, i.e. $a^*=b^* \rightarrow a \equiv b$. Four cases for $a^*=b^*$ need to be tackled here, which correspond to the four possibilities for a and b: zilch, V_1, V_2, any other element. Supposing $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle = \langle \mathbf{y}_1, \mathbf{y}_2 \rangle$ it follows that $\mathbf{x}_1^* \neq \mathbf{x}_2^*$ and $\mathbf{y}_1^* \neq \mathbf{y}_2^*$ which in turn entail $\mathbf{x}_1^* = \mathbf{y}_1^*$ and $\mathbf{x}_2^* = \mathbf{y}_2^*$. Since $a^*=b^* \rightarrow a \equiv b$, it follows that $\mathbf{x}_1 \equiv \mathbf{y}_1 \wedge \mathbf{x}_2 \equiv \mathbf{y}_2$.The proof of the $\leftarrow$ half proceeds by showing that different items code different coordinates, i.e. $a \equiv b \rightarrow a^*=b^*$. Supposing $\mathbf{x}_1 \equiv \mathbf{y}_1 \wedge \mathbf{x}_2 \equiv \mathbf{y}_2$, it follows that $|\mathbf{x}_1^*, \mathbf{x}_2^*| \equiv |\mathbf{y}_1^*, \mathbf{y}_2^*|$, i.e. $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle \equiv \langle \mathbf{y}_1, \mathbf{y}_2 \rangle$, whence $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle = \langle \mathbf{y}_1, \mathbf{y}_2 \rangle$ by (i).

THEOREM 39. *Existence of classes*

(i) *Let* $m\mathbf{x}U\mathbf{x}$, *then* $\mathsf{E}!(\mathbf{x} \cdot \mathbf{x} \notin \mathbf{x})$.

(ii) *Let* $\mathsf{E}!(\mathbf{x} \cdot \mathbf{x} \notin \mathbf{x})$, *then* $C(\mathbf{x} \cdot \mathbf{x} \notin \mathbf{x})$.

(iii) $m\mathbf{x}U\mathbf{x} \leftrightarrow \exists \mathbf{x} C\mathbf{x}$

PROOF OF (i). By the hypothesis and axiom 2(i) it follows that $m\mathbf{x}(E\mathbf{x} \wedge U\mathbf{x})$, whence $m\mathbf{x}(E\mathbf{x} \wedge \mathbf{x} \notin \mathbf{x})$ by axiom 1(iv) and the definition of U. Hence $\mathsf{E}!(\mathbf{x} \cdot \mathbf{x} \notin \mathbf{x})$ by lemma 2(i).

PROOF OF (ii). By the hypothesis it follows that $M(\mathbf{x}\cdot\mathbf{x}\notin\mathbf{x}) \wedge \forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\cdot\mathbf{x}\notin\mathbf{x} \leftrightarrow (E\mathbf{y} \wedge \mathbf{y}\notin\mathbf{y}))$ by lemma 4, whence $S(\mathbf{x}\cdot\mathbf{x}\notin\mathbf{x}) \vee C(\mathbf{x}\cdot\mathbf{x}\notin\mathbf{x})$ by lemma 3(ii). For a reductio suppose $S(\mathbf{x}\cdot\mathbf{x}\notin\mathbf{x})$. Then $E(\mathbf{x}\cdot\mathbf{x}\notin\mathbf{x})$ by the definition of S, whence $(\mathbf{x}\cdot\mathbf{x}\notin\mathbf{x} \in \mathbf{x}\cdot\mathbf{x}\notin\mathbf{x})\leftrightarrow(\mathbf{x}\cdot\mathbf{x}\notin\mathbf{x} \notin \mathbf{x}\cdot\mathbf{x}\notin\mathbf{x})$. Contradiction. Hence $C(\mathbf{x}\cdot\mathbf{x}\notin\mathbf{x})$.

PROOF OF (iii).
1 For the $\rightarrow$ half, suppose $m\mathbf{x}U\mathbf{x}$. Then $\exists\mathbf{x}C\mathbf{x}$ by theorems 39(i) and 39(ii).
2 For the $\leftarrow$ half, suppose $\exists\mathbf{x}C\mathbf{x}$. Then $m\mathbf{x}U\mathbf{x}$ by the definition of C and axiom 2(ii).

THEOREM 40. *Classes and non-self-membership*
Let $\mathsf{E}!(\mathbf{x}\cdot A(\mathbf{x})) \wedge S(\mathbf{x}\cdot A(\mathbf{x}))\rightarrow A(\mathbf{x}\cdot A(\mathbf{x}))$, *then* $C(\mathbf{x}\cdot A(\mathbf{x}))$.

PROOF. By the hypothesis and lemma 4 it follows that $M(\mathbf{x}\cdot A(\mathbf{x})) \wedge \forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\cdot A(\mathbf{x})\leftrightarrow(E\mathbf{y} \wedge A(\mathbf{y})))$, whence $S(\mathbf{x}\cdot A(\mathbf{x})) \vee C(\mathbf{x}\cdot A(\mathbf{x}))$ by lemma 3(ii). For a reductio suppose $S(\mathbf{x}\cdot A(\mathbf{x}))$. Then by the hypothesis and the definition of S, $\mathbf{x}\cdot A(\mathbf{x})\in\mathbf{x}\cdot A(\mathbf{x})$ contrary to axiom 1(iii). Hence $\neg S(\mathbf{x}\cdot A(\mathbf{x}))$, whence $C(\mathbf{x}\cdot A(\mathbf{x}))$.

THEOREM 41. *Classes and levels*
(i) $C\mathbf{x}\leftrightarrow(M\mathbf{x} \wedge \neg\exists\mathbf{u}\ \mathbf{x}\subseteq\mathbf{u})$
(ii) $C\mathbf{x}\leftrightarrow(M\mathbf{x} \wedge \forall\mathbf{u}\exists\mathbf{y}(\mathbf{y}\in\mathbf{x} \wedge \mathbf{u}\in V^*(\mathbf{y}))$
(iii) $C\mathbf{x}\leftrightarrow(M\mathbf{x} \wedge \neg\exists\mathbf{u}\ \mathbf{x}\in\mathbf{u})$
(iv) *Let* $M\mathbf{x} \wedge \forall\mathbf{u}\exists\mathbf{y}(\mathbf{y}\in\mathbf{x} \wedge (\mathbf{u}\in\mathbf{y} \vee \mathbf{u}\subseteq\mathbf{y}))$, *then* $C\mathbf{x}$.

PROOF OF (i). By the definitions of C and S, $C\mathbf{x}\leftrightarrow(M\mathbf{x} \wedge \neg S\mathbf{x})$, whence by theorem 1(ii), $C\mathbf{x}\leftrightarrow(M\mathbf{x} \wedge \neg\exists\mathbf{u}\ \mathbf{x}\subseteq\mathbf{u})$.

PROOF OF (ii).
1 For the $\rightarrow$ half, suppose $C\mathbf{x}$. Then $M\mathbf{x}$ by the definition of C. Hence by lemma 3(i) $\mathsf{E}!\mathbf{x}$, whence by theorems 28(i) and 28(ii), $\exists\mathbf{z}V\mathbf{z}$. Consider an arbitrary level $\mathbf{v}$. Then $\mathsf{E}!\mathbf{v}'$ by theorem 30, whence $V\mathbf{v}' \wedge \mathbf{v}\in\mathbf{v}'$ by the definitions of $\mathbf{v}'$ and $V^\dagger(\mathbf{v})$.
2 For a reductio suppose $\neg\exists\mathbf{y}(\mathbf{y}\in\mathbf{x} \wedge \mathbf{v}\in V^*(\mathbf{y}))$. Since $M\mathbf{x}$, it follows that $m\mathbf{z}\ \mathbf{z}\in\mathbf{x}$ by the definition of M. Consider an arbitrary $\mathbf{y}_1\in\mathbf{x}$. Then by lemmas 3(iii) and 5(i) $U\mathbf{y}_1 \vee S\mathbf{y}_1$. Suppose $U\mathbf{y}_1$. Then $\mathbf{y}_1\in\mathbf{v}'$ by lemma 8(iv). Suppose instead $S\mathbf{y}_1$. Then $V^*(\mathbf{y}_1)\in\mathbf{v} \vee \mathbf{v}=V^*(\mathbf{y}_1)$ by theorems 14 and 17(i), and the definition of $V^*(\mathbf{y}_1)$. Suppose $V^*(\mathbf{y}_1)\in\mathbf{v}$. Then $\mathbf{y}_1\in\mathbf{v}$ by the definition of $V^*(\mathbf{y}_1)$ and theorem 9, whence $\mathbf{y}_1\in\mathbf{v}'$ by theorem 8. Suppose instead $\mathbf{v}=V^*(\mathbf{y}_1)$. Then $\mathbf{v}'=(V^*(\mathbf{y}_1))'$, whence $\mathbf{y}_1\in\mathbf{v}'$ by theorem 32 and the definition of $V^\dagger(\mathbf{y}_1)$. So either way $\mathbf{y}_1\in\mathbf{v}'$. Since $\mathbf{y}_1$ is arbitrary, we can generalize to get $\forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\rightarrow\mathbf{y}\in\mathbf{v}')$, whence $\exists\mathbf{u}\ \mathbf{x}\subseteq\mathbf{u}$ by the definitions of $\subseteq$. But by theorem 41(i) $\neg\exists\mathbf{u}\ \mathbf{x}\subseteq\mathbf{u}$. Contradiction. Hence $\exists\mathbf{y}(\mathbf{y}\in\mathbf{x} \wedge \mathbf{v}\in V^*(\mathbf{y}))$. Since $\mathbf{v}$ is arbitrary, we can generalize to get $\forall\mathbf{u}\exists\mathbf{y}(\mathbf{y}\in\mathbf{x} \wedge \mathbf{u}\in V^*(\mathbf{y}))$.
3 For the $\leftarrow$ half, suppose $M\mathbf{x} \wedge \forall\mathbf{u}\exists\mathbf{y}(\mathbf{y}\in\mathbf{x} \wedge \mathbf{u}\in V^*(\mathbf{y}))$. Then by lemma 3(i) and theorems 28(i) and 28(ii), $\exists\mathbf{z}V\mathbf{z}$. Consider an arbitrary level $\mathbf{v}$. Then for some $\mathbf{y}$, $S\mathbf{y} \wedge \mathbf{y}\in\mathbf{x} \wedge \mathbf{v}\in V^*(\mathbf{y})$ by axiom 1(i) and theorem 17(i).
4 For a reductio suppose $V^*(\mathbf{y})\in\mathbf{v}$. Then by theorem 8, $\mathbf{v}\in\mathbf{v}$, contrary to axiom 1(iii), whence $\exists\mathbf{y}(S\mathbf{y} \wedge \mathbf{y}\in\mathbf{x} \wedge V^*(\mathbf{y})\notin\mathbf{v})$. Since $\mathbf{v}$ is arbitrary, we can generalize to get

$\forall \mathbf{u}\exists \mathbf{y}(S\mathbf{y} \wedge \mathbf{y}\in\mathbf{x} \wedge V^*(\mathbf{y})\notin\mathbf{u})$, whence $\neg\exists \mathbf{u}\forall \mathbf{y}((S\mathbf{y} \wedge \mathbf{y}\in\mathbf{x})\to V^*(\mathbf{y})\in\mathbf{u})$. Hence by theorem 17(vi), $\neg S\mathbf{x}$, whence $C\mathbf{x}$ by lemma 3(ii).

PROOF OF (iii).

1 For the $\to$ half, suppose $C\mathbf{x}$. Then $M\mathbf{x} \wedge \neg\exists \mathbf{u}\ \mathbf{x}\in\mathbf{u}$ by the definitions of C and E.

2 For the $\leftarrow$ half, suppose $M\mathbf{x} \wedge \neg\exists \mathbf{u}\ \mathbf{x}\in\mathbf{u}$. Then $S\mathbf{x} \vee C\mathbf{x}$ by lemma 3(ii). But $\neg S\mathbf{x}$ by theorem 22(i), whence $C\mathbf{x}$.

PROOF OF (iv). For a reductio suppose $\mathbf{x}\in\mathbf{v}$ for some level $\mathbf{v}$. Then by the hypothesis, for some $\mathbf{y}$, $\mathbf{y}\in\mathbf{x} \wedge (\mathbf{v}\in\mathbf{y} \vee \mathbf{v}\subseteq\mathbf{y}))$, whence $\mathbf{y}\in\mathbf{v}$ by theorem 8. Suppose $\mathbf{v}\in\mathbf{y}$. Then by theorem 8, $\mathbf{v}\in\mathbf{v}$. Suppose $\mathbf{v}\subseteq\mathbf{y}$. Then by theorem 9, $\mathbf{v}\in\mathbf{v}$. So either way $\mathbf{v}\in\mathbf{v}$, contrary to axiom 1(iii). Hence $\neg\exists \mathbf{u}\ \mathbf{x}\in\mathbf{u}$, whence $C\mathbf{x}$ by the hypothesis and theorem 41(iii).

THEOREM 42. *Classes: an illustrative sample*

(i) *Let* $\mathsf{E}!(\mathbf{x}\cdot\mathbf{y}\in\mathbf{x})$, *then* $C(\mathbf{x}\cdot\mathbf{y}\in\mathbf{x})$.

(ii) *Let* $\mathsf{E}!(\mathbf{x}\cdot\mathbf{y}\notin\mathbf{x})$, *then* $C(\mathbf{x}\cdot\mathbf{y}\notin\mathbf{x})$.

(iii) *Let* $\mathsf{E}!(\mathbf{x}\cdot\mathbf{x}\notin\mathbf{y})$ *and* $\neg C\mathbf{y}$, *then* $C(\mathbf{x}\cdot\mathbf{x}\notin\mathbf{y})$.

(iv) *Let* $\mathsf{E}!(\mathbf{x}\cdot\mathbf{x}\neq\mathbf{y})$, *then* $C(\mathbf{x}\cdot\mathbf{x}\neq\mathbf{y})$.

(v) *Let* $\mathsf{E}!(\mathbf{x}\cdot\mathbf{y}\subseteq\mathbf{x})$, *then* $C(\mathbf{x}\cdot\mathbf{y}\subseteq\mathbf{x})$.

(vi) *Let* $\mathsf{E}!(\mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=P(\mathbf{y}))$, *then* $C(\mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=P(\mathbf{y}))$.

(vii) *Let* $\mathsf{E}!(\mathbf{x}\cdot\exists\mathbf{y}\exists\mathbf{z}\ \mathbf{x}=|\mathbf{y},\mathbf{z}|)$, *then* $C(\mathbf{x}\cdot\exists\mathbf{y}\exists\mathbf{z}\ \mathbf{x}=|\mathbf{y},\mathbf{z}|)$.

(viii) *Let* $\mathsf{E}!(\mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=\cup\mathbf{y})$, *then* $C(\mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=\cup\mathbf{y})$.

(ix) *Let* $\mathsf{E}!(\mathbf{x}\cdot H\mathbf{x})$, *then* $C(\mathbf{x}\cdot H\mathbf{x})$.

(x) *Let* $\mathsf{E}!(\mathbf{x}\cdot V\mathbf{x})$, *then* $C(\mathbf{x}\cdot V\mathbf{x})$.

PROOF OF (i).

1 By the hypothesis and lemma 4, it follows that $M(\mathbf{x}\cdot\mathbf{y}\in\mathbf{x}) \wedge \forall\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\mathbf{y}\in\mathbf{x}\leftrightarrow(E\mathbf{z} \wedge \mathbf{y}\in\mathbf{z})$, whence $\exists\mathbf{x}V\mathbf{x}$ by theorems 28(i) and 28(ii). By the hypothesis and lemma 2(i), $E\mathbf{x}_1 \wedge \mathbf{y}\in\mathbf{x}_1$ for some $\mathbf{x}_1$, whence $M\mathbf{x}_1 \wedge S\mathbf{x}_1$ by lemma 5(i) and the definition of S, and also $E\mathbf{y}$ by the definition of E.

2 Consider an arbitrary level $\mathbf{v}$. Then $E\mathbf{v}$ by lemma 8(ii), whence $S(\mathbf{x}_1\oplus\mathbf{v}) \wedge \mathsf{E}!(\mathbf{x}_1\oplus\mathbf{v}) \wedge E(\mathbf{x}_1\oplus\mathbf{v})$ by theorem 24(ii), lemmas 3(i) and 3(ii), and the definition of S. Hence $\mathbf{y}\in\mathbf{x}_1\oplus\mathbf{v} \wedge \mathbf{v}\in\mathbf{x}_1\oplus\mathbf{v}$ by lemma 4 and the definition of $\mathbf{x}_1\oplus\mathbf{v}$, whence $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\mathbf{y}\in\mathbf{x} \wedge \mathbf{v}\in\mathbf{z})$. Since $\mathbf{v}$ is arbitrary, we can generalize to get $\forall\mathbf{u}\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\mathbf{y}\in\mathbf{x} \wedge (\mathbf{u}\in\mathbf{z} \vee \mathbf{u}\subseteq\mathbf{z}))$, whence $C(\mathbf{x}\cdot\mathbf{y}\in\mathbf{x})$ by theorem 41(iv).

PROOF OF (ii).

1 By the hypothesis and lemma 4, it follows that $M(\mathbf{x}\cdot\mathbf{y}\notin\mathbf{x}) \wedge \forall\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\mathbf{y}\notin\mathbf{x}\leftrightarrow(E\mathbf{z} \wedge \mathbf{y}\notin\mathbf{z})$, whence $m\mathbf{x}U\mathbf{x} \wedge \exists\mathbf{x}V\mathbf{x}$ by theorems 28(i) and 28(ii). Consider an arbitrary level $\mathbf{v}$. Then $E\mathbf{v}$ by lemma 8(ii), and $\mathbf{v}\subseteq\mathbf{v}$ by lemmas 6(ii) and 8(i).

2 Suppose $\mathbf{y}=\mathbf{v}$. Then $\mathbf{y}\notin\mathbf{v}$ by axiom 1(iii). Hence $\mathbf{v}\in\mathbf{x}\cdot\mathbf{y}\notin\mathbf{x}$, whence $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\mathbf{y}\notin\mathbf{x} \wedge (\mathbf{v}\in\mathbf{z} \vee \mathbf{v}\subseteq\mathbf{z}))$. Suppose instead $\mathbf{y}\neq\mathbf{v}$. Since $m\mathbf{x}U\mathbf{x}$, it follows that for some $\mathbf{z}_1$, $U\mathbf{z}_1 \wedge \mathbf{y}\neq\mathbf{z}_1$, whence $E\mathbf{z}_1 \wedge \mathbf{v}\neq\mathbf{z}_1$ by lemmas 3(iii) and 8(i), and the definition of U. Hence $\mathsf{E}!|\mathbf{z}_1,\mathbf{v}| \wedge E|\mathbf{z}_1,\mathbf{v}|$ by theorems 23(i) and 23(ii), and the definition of S. Hence by

lemma 4 and the definition of $|\mathbf{z_1}, \mathbf{v}|$, $\mathbf{y}\notin|\mathbf{z_1}, \mathbf{v}| \wedge \mathbf{v}\in|\mathbf{z_1}, \mathbf{v}|$, whence $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\mathbf{y}\notin\mathbf{x} \wedge (\mathbf{v}\in\mathbf{z} \vee \mathbf{v}\subseteq\mathbf{z}))$. So either way, $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\mathbf{y}\notin\mathbf{x} \wedge (\mathbf{v}\in\mathbf{z} \vee \mathbf{v}\subseteq\mathbf{z}))$. Since **v** is arbitrary, we can generalize to get $\forall\mathbf{u}\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\mathbf{y}\notin\mathbf{x} \wedge (\mathbf{u}\in\mathbf{z} \vee \mathbf{u}\subseteq\mathbf{z}))$, whence $\mathrm{C}(\mathbf{x}\cdot\mathbf{y}\notin\mathbf{x})$ by theorem 41(iv).

PROOF OF (iii).

1 By the hypothesis and lemma 4, it follows that $M(\mathbf{x}\cdot\mathbf{x}\notin\mathbf{y}) \wedge \forall\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\mathbf{x}\notin\mathbf{y} \leftrightarrow (E\mathbf{z} \wedge \mathbf{z}\notin\mathbf{y})$, whence $\exists\mathbf{x}V\mathbf{x}$ by theorems 28(i) and 28(ii). Consider an arbitrary level **v**. Then $E\mathbf{v}$ by lemma 8(ii), and $\mathbf{v}\subseteq\mathbf{v}$ by lemmas 6(ii) and 8(i).

2 Suppose $\mathbf{v}\notin\mathbf{y}$. Then $\mathbf{v}\in\mathbf{x}\cdot\mathbf{x}\notin\mathbf{y}$, whence $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\mathbf{x}\notin\mathbf{y} \wedge (\mathbf{v}\in\mathbf{z} \vee \mathbf{v}\subseteq\mathbf{z}))$. Suppose instead $\mathbf{v}\in\mathbf{y}$. Then $\mathsf{E}!\mathbf{y} \wedge E\mathbf{y} \wedge \mathbf{y}\notin\mathbf{y}$ by axioms 1(i), 1(iii) and 1(iv), the hypothesis and the definition of C. Hence $\mathbf{y}\in\mathbf{x}\cdot\mathbf{x}\notin\mathbf{y}$, whence $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\mathbf{x}\notin\mathbf{y} \wedge (\mathbf{v}\in\mathbf{z} \vee \mathbf{v}\subseteq\mathbf{z}))$. So either way, $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\mathbf{x}\notin\mathbf{y} \wedge (\mathbf{v}\in\mathbf{z} \vee \mathbf{v}\subseteq\mathbf{z}))$. Since **v** is arbitrary, we can generalize to get $\forall\mathbf{u}\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\mathbf{x}\notin\mathbf{y} \wedge (\mathbf{u}\in\mathbf{z} \vee \mathbf{u}\subseteq\mathbf{z}))$, whence $C(\mathbf{x}\cdot\mathbf{x}\notin\mathbf{y})$ by theorem 41(iv).

PROOF OF (iv).

1 By the hypothesis and lemma 4, it follows that $M(\mathbf{x}\cdot\mathbf{x}\neq\mathbf{y}) \wedge \forall\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\mathbf{x}\neq\mathbf{y} \leftrightarrow (E\mathbf{z} \wedge \mathbf{z}\neq\mathbf{y})$, whence $\exists\mathbf{x}V\mathbf{x}$ by theorems 28(i) and 28(ii). Consider an arbitrary level **v**. Then $E\mathbf{v}$ by lemma 8(ii), and $\mathbf{v}\subseteq\mathbf{v}$ by lemmas 6(ii) and 8(i).

2 Suppose $\mathbf{v}\neq\mathbf{y}$. Then $\mathbf{v}\in\mathbf{x}\cdot\mathbf{x}\neq\mathbf{y}$, whence $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\mathbf{x}\neq\mathbf{y} \wedge (\mathbf{v}\in\mathbf{z} \vee \mathbf{v}\subseteq\mathbf{z}))$. Suppose instead $\mathbf{v}=\mathbf{y}$. Then by theorem 30, the definitions of $\mathbf{v}'$ and $V^{\dagger}(\mathbf{v})$, axiom 1(iii) and lemma 8(ii), $\mathsf{E}!\mathbf{v}' \wedge E\mathbf{v}' \wedge \mathbf{v}'\neq\mathbf{y} \wedge \mathbf{v}\in\mathbf{v}'$. Hence $\mathbf{v}'\in\mathbf{x}\cdot\mathbf{x}\neq\mathbf{y}$, whence $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\mathbf{x}\neq\mathbf{y} \wedge (\mathbf{v}\in\mathbf{z} \vee \mathbf{v}\subseteq\mathbf{z}))$. So either way, $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\mathbf{x}\neq\mathbf{y} \wedge (\mathbf{v}\in\mathbf{z} \vee \mathbf{v}\subseteq\mathbf{z}))$. Since **v** is arbitrary, we can generalize to get $\forall\mathbf{u}\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\mathbf{x}\neq\mathbf{y} \wedge (\mathbf{u}\in\mathbf{z} \vee \mathbf{u}\subseteq\mathbf{z}))$, whence $C(\mathbf{x}\cdot\mathbf{x}\neq\mathbf{y})$ by theorem 41(iv).

PROOF OF (v).

1 By the hypothesis and lemma 4, it follows that $M(\mathbf{x}\cdot\mathbf{y}\subseteq\mathbf{x}) \wedge \forall\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\mathbf{y}\subseteq\mathbf{x} \leftrightarrow (E\mathbf{z} \wedge \mathbf{y}\subseteq\mathbf{z})$, whence $\exists\mathbf{x}V\mathbf{x}$ by theorems 28(i) and 28(ii). By the hypothesis and lemma 2(i), $E\mathbf{x_1} \wedge \mathbf{y}\subseteq\mathbf{x_1}$ for some $\mathbf{x_1}$, whence $M\mathbf{y} \wedge S\mathbf{y}$ by lemma 6(i), the definition of S, and axiom 2(iii).

2 Consider an arbitrary level **v**. Then $M\mathbf{v} \wedge S\mathbf{v}$ by lemmas 8(i) and 8(iii), whence $S(\mathbf{y}\cup\mathbf{v}) \wedge E(\mathbf{y}\cup\mathbf{v}) \wedge \mathsf{E}!(\mathbf{y}\cup\mathbf{v})$ by lemmas 3(i) and 3(ii), theorem 20(ii) and the definition of S. Hence $\mathbf{y}\subseteq\mathbf{y}\cup\mathbf{v} \wedge \mathbf{v}\subseteq\mathbf{y}\cup\mathbf{v}$ by lemmas 4 and 5(i), and the definitions of $\mathbf{y}\cup\mathbf{v}$ and $\subseteq$. Hence $\mathbf{y}\cup\mathbf{v} \in \mathbf{x}\cdot\mathbf{y}\subseteq\mathbf{x}$, whence $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\mathbf{y}\subseteq\mathbf{x} \wedge (\mathbf{v}\in\mathbf{z} \vee \mathbf{v}\subseteq\mathbf{z}))$. Since **v** is arbitrary, we can generalize to get $\forall\mathbf{u}\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\mathbf{y}\subseteq\mathbf{x} \wedge (\mathbf{u}\in\mathbf{z} \vee \mathbf{u}\subseteq\mathbf{z}))$, whence $C(\mathbf{x}\cdot\mathbf{y}\subseteq\mathbf{x})$ by theorem 41(iv).

PROOF OF (vi).

1 By the hypothesis and lemma 4, it follows that $M(\mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=P(\mathbf{y})) \wedge \forall\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=P(\mathbf{y}) \leftrightarrow (E\mathbf{z} \wedge \exists\mathbf{y}\ \mathbf{z}=P(\mathbf{y})))$, whence $m\mathbf{x}U\mathbf{x} \wedge \exists\mathbf{x}V\mathbf{x}$ by theorems 28(i) and 28(ii). Consider an arbitrary level **v**. Then $\mathbf{v}\subseteq\mathbf{v}$ by lemmas 6(ii) and 8(i), and also $\mathsf{E}!P(\mathbf{v}) \wedge E(P(\mathbf{v}))$ by lemmas 3(i), 3(ii) and 8(iii), theorem 25(iv), and the definition of S, whence $P(\mathbf{v})\in \mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=P(\mathbf{y})$.

2 Suppose $\exists\mathbf{y_1}\exists\mathbf{y_2}\exists\mathbf{y_3}(U\mathbf{y_1} \wedge U\mathbf{y_2} \wedge U\mathbf{y_3} \wedge \mathbf{y_1}\neq\mathbf{y_2} \wedge \mathbf{y_1}\neq\mathbf{y_3} \wedge \mathbf{y_2}\neq\mathbf{y_3}) \vee (\exists\mathbf{y_1}\exists\mathbf{y_2}(U\mathbf{y_1} \wedge U\mathbf{y_2} \wedge \mathbf{y_1}\neq\mathbf{y_2} \wedge \forall\mathbf{y_3}(U\mathbf{y_3}\rightarrow (\mathbf{y_3}=\mathbf{y_1} \vee \mathbf{y_3}=\mathbf{y_2}))) \wedge \mathbf{v}\neq V_1)$. Then $P(\mathbf{v})=\mathbf{y}\cdot\mathbf{y}\subseteq\mathbf{v}$ by theorems 33(i) and 33(ii), whence $\mathbf{v}\in P(\mathbf{v})$ by lemmas 4 and 8(ii). Hence $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=P(\mathbf{y}) \wedge (\mathbf{v}\in\mathbf{z} \vee$

$\mathbf{v}\subseteq\mathbf{z})$). Suppose instead $\exists\mathbf{y}_1\exists\mathbf{y}_2(U\mathbf{y}_1 \wedge U\mathbf{y}_2 \wedge \mathbf{y}_1\neq\mathbf{y}_2 \wedge \forall\mathbf{y}_3(U\mathbf{y}_3\rightarrow (\mathbf{y}_3=\mathbf{y}_1 \vee \mathbf{y}_3=\mathbf{y}_2))) \wedge \mathbf{v}=V_1)$. Then $P(\mathbf{v})=\mathbf{v}$ by theorem 33(i), whence $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=P(\mathbf{y}) \wedge (\mathbf{v}\in\mathbf{z} \vee \mathbf{v}\subseteq\mathbf{z}))$. So either way $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=P(\mathbf{y}) \wedge (\mathbf{v}\in\mathbf{z} \vee \mathbf{v}\subseteq\mathbf{z}))$. Since **v** is arbitrary, we can generalize to get $\forall\mathbf{u}\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=P(\mathbf{y}) \wedge (\mathbf{u}\in\mathbf{z} \vee \mathbf{u}\subseteq\mathbf{z}))$, whence $C(\mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=P(\mathbf{y}))$ by theorem 41(iv).

PROOF OF (vii).

1 By the hypothesis and lemma 4, it follows that $M(\mathbf{x}\cdot\exists\mathbf{y}\exists\mathbf{z}\ \mathbf{x}=|\mathbf{y},\mathbf{z}|) \wedge \forall\mathbf{z}_1(\mathbf{z}_1\in\mathbf{x}\cdot\exists\mathbf{y}\exists\mathbf{z}\ \mathbf{x}=|\mathbf{y},\mathbf{z}|\leftrightarrow(E\mathbf{z}_1 \wedge \exists\mathbf{y}\exists\mathbf{z}\ \mathbf{z}_1=|\mathbf{y},\mathbf{z}|))$, whence $\exists\mathbf{x}V\mathbf{x}$ by theorems 28(i) and 28(ii). Consider an arbitrary level **v**. Then $E\mathbf{v} \wedge \mathsf{E}!\mathbf{v}' \wedge E\mathbf{v}'$ by theorem 30, the definitions of $\mathbf{v}'$ and $V^{\dagger}(\mathbf{v})$, and lemma 8(ii). Also $\mathbf{v}\neq\mathbf{v}'$ by the definitions of $\mathbf{v}'$ and $V^{\dagger}(\mathbf{v})$, and axiom 1(iii). Hence $\mathsf{E}!|\mathbf{v},\mathbf{v}'| \wedge E|\mathbf{v},\mathbf{v}'|$ by theorems 23(i) and 23(ii), and the definition of S, whence $|\mathbf{v},\mathbf{v}'| \in \mathbf{x}\cdot\exists\mathbf{y}\exists\mathbf{z}\ \mathbf{x}=|\mathbf{y},\mathbf{z}|)$.

2 By lemma 4 and the definition of $|\mathbf{v},\mathbf{v}'|$, it follows that $\mathbf{v}\in|\mathbf{v},\mathbf{v}'|$, whence $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\exists\mathbf{y}\exists\mathbf{z}\ \mathbf{x}=|\mathbf{y},\mathbf{z}| \wedge (\mathbf{v}\in\mathbf{z} \vee \mathbf{v}\subseteq\mathbf{z}))$. Since **v** is arbitrary, we can generalize to get $\forall\mathbf{u}\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\exists\mathbf{y}\exists\mathbf{z}\ \mathbf{x}=|\mathbf{y},\mathbf{z}| \wedge (\mathbf{u}\in\mathbf{z} \vee \mathbf{u}\subseteq\mathbf{z}))$, whence $C(\mathbf{x}\cdot\exists\mathbf{y}\exists\mathbf{z}\ \mathbf{x}=|\mathbf{y},\mathbf{z}|)$ by theorem 41(iv).

PROOF OF (viii).

1 By the hypothesis and lemma 4, it follows that $M(\mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=\cup\mathbf{y}) \wedge \forall\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=\cup\mathbf{y} \leftrightarrow(E\mathbf{z} \wedge \exists\mathbf{y}\ \mathbf{z}=\cup\mathbf{y}))$, whence $\exists\mathbf{x}V\mathbf{x}$ by theorems 28(i) and 28(ii). Consider an arbitrary level **v**. Then $M\mathbf{v} \wedge M\mathbf{v}' \wedge S\mathbf{v} \wedge S\mathbf{v}' \wedge E\mathbf{v} \wedge E\mathbf{v}'$ by theorem 30, the definitions of $\mathbf{v}'$ and $V^{\dagger}(\mathbf{v})$, and lemmas 8(i), 8(ii) and 8(iii). Also $\mathbf{v}\neq\mathbf{v}'$ by the definition of $\mathbf{v}'$ and $V^{\dagger}(\mathbf{v})$, and axiom 1(iii), whence $\mathsf{E}!|\mathbf{v},\mathbf{v}'| \wedge S|\mathbf{v},\mathbf{v}'|$ by theorems 23(i) and 23(ii).

2 By lemma 4 and the definition of $|\mathbf{v},\mathbf{v}'|$, it follows that $\mathbf{v}\in|\mathbf{v},\mathbf{v}'| \wedge \forall\mathbf{z}(\mathbf{z}\in|\mathbf{v},\mathbf{v}'|\rightarrow M\mathbf{z}) \wedge \forall\mathbf{z}(\mathbf{z}\in|\mathbf{v},\mathbf{v}'|\rightarrow S\mathbf{z})$. Hence $\mathsf{E}!\cup|\mathbf{v},\mathbf{v}'| \wedge E\cup|\mathbf{v},\mathbf{v}'|$ by corollary (ii) of theorem 21, lemmas 3(i) and 3(ii), and the definition of S, whence $\cup|\mathbf{v},\mathbf{v}'| \in \mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=\cup\mathbf{y}$. Also $\forall\mathbf{z}(\mathbf{z}\in\mathbf{v}\rightarrow\mathbf{z}\in\cup|\mathbf{v},\mathbf{v}'|)$ by lemmas 4 and 5(i), and the definition of $\cup|\mathbf{v},\mathbf{v}'|$. Hence $\mathbf{v}\subseteq\cup|\mathbf{v},\mathbf{v}'|$ by the definition of $\subseteq$, whence $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=\cup\mathbf{y} \wedge (\mathbf{v}\in\mathbf{z} \vee \mathbf{v}\subseteq\mathbf{z}))$. Since **v** is arbitrary, we can generalize to get $\forall\mathbf{u}\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=\cup\mathbf{y} \wedge (\mathbf{u}\in\mathbf{z} \vee \mathbf{u}\subseteq\mathbf{z}))$, whence $C(\mathbf{x}\cdot\exists\mathbf{y}\ \mathbf{x}=\cup\mathbf{y})$ by theorem 41(iv).

PROOF OF (ix).

1 By the hypothesis and lemma 4, it follows that $M(\mathbf{x}\cdot H\mathbf{x}) \wedge \forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\cdot H\mathbf{x}\leftrightarrow(E\mathbf{y} \wedge H\mathbf{y}))$, whence $\exists\mathbf{x}V\mathbf{x}$ by theorems 28(i) and 28(ii). Consider an arbitrary level **v**. Then $V(\mathbf{v}'') \wedge \mathbf{v}\in\mathbf{v}' \wedge \mathbf{v}'\in\mathbf{v}'' \wedge \mathbf{v}\in\mathbf{v}''$ by theorems 8 and 30, and the definitions of $\mathbf{v}'$, $\mathbf{v}''$, $V^{\dagger}(\mathbf{v})$ and $V^{\dagger}(\mathbf{v}')$, whence $\mathsf{E}!\mathbf{w}{:}\mathbf{w}\in\mathbf{v}''$ by lemmas 2(iv) and 8(ii). Hence $H(\mathbf{w}{:}\mathbf{w}\in\mathbf{v}'') \wedge E(\mathbf{w}{:}\mathbf{w}\in\mathbf{v}'')$ by lemma 7(ix), theorem 12, and the definitions of H and S, whence $\mathbf{w}{:}\mathbf{w}\in\mathbf{v}''\in \mathbf{x}\cdot H\mathbf{x}$.

2 Since $\mathbf{v}\in\mathbf{v}' \wedge \mathbf{v}'\in\mathbf{v}'' \wedge \mathbf{v}\in\mathbf{v}''$, it follows by axioms 1(i) and 1(iii) that $m\mathbf{w}\ \mathbf{w}\in\mathbf{v}''$, whence by lemmas 2(ii) and 8(ii), $\mathbf{w}{:}\mathbf{w}\in\mathbf{v}'' = \mathbf{w}\cdot\mathbf{w}\in\mathbf{v}''$. Hence by lemmas 4 and 8(ii), $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot H\mathbf{x} \wedge (\mathbf{v}\in\mathbf{z} \vee \mathbf{v}\subseteq\mathbf{z}))$. Since **v** is arbitrary, we can generalize to get $\forall\mathbf{u}\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot H\mathbf{x} \wedge (\mathbf{u}\in\mathbf{z} \vee \mathbf{u}\subseteq\mathbf{z}))$, whence $C\mathbf{x}\cdot H\mathbf{x}$ by theorem 41(iv).

PROOF OF (x). By the hypothesis and lemma 4 it follows that $M(\mathbf{x}\cdot V\mathbf{x}) \wedge \forall\mathbf{y}(\mathbf{y}\in\mathbf{x}\cdot V\mathbf{x} \leftrightarrow (E\mathbf{y} \wedge V\mathbf{y}))$, whence $\exists\mathbf{x}V\mathbf{x}$ by theorems 28(i) and 28(ii). Consider an arbitrary level **v**. Then $E\mathbf{v}$ by lemma 8(ii), and $\mathbf{v}\subseteq\mathbf{v}$ by lemmas 6(ii) and 8(i). Hence $\mathbf{v}\in\mathbf{x}\cdot V$x, whence $\exists\mathbf{z}(\mathbf{z}\in\mathbf{x}\cdot V\mathbf{x}$

$\wedge$ (**v**$\in$**z** $\vee$ **v**$\subseteq$**z**)). Since **v** is arbitrary, we can generalize to get $\forall$**u**$\exists$**z**(**z**$\in$**x**$\cdot$*V***x** $\wedge$ (**u**$\in$**z** $\vee$ **u**$\subseteq$**z**)), whence *C*(**x**$\cdot$*V***x**) by theorem 41(iv).

THEOREM 43. *Classes and separation*
(i) *Let C***x** $\wedge$ **x**$\subseteq$**y**, *then C***y**.
(ii) *Let C*(**z**$\cdot$**z**$\in$**x** $\wedge$ *A*(**z**)), *then C***x**.

PROOF OF (i). By the hypothesis and the definitions of *C* and *S*, it follows that $\neg$*S***x**, whence $\neg$*S***y** by axiom 2(iii). Since *M***y** by the hypothesis and lemma 6(i), it follows that *C***y** by lemma 3(ii).

PROOF OF (ii). By the hypothesis and the definitions of *C* and *S*, it follows that $\neg$*S*(**z**$\cdot$**z**$\in$**x** $\wedge$ *A*(**z**)), and also that E!(**z**$\cdot$**z**$\in$**x** $\wedge$ *A*(**z**)) by lemmas 3(i) and 3(ii). Hence *m***y**(**y**$\in$**x** $\wedge$ *A*(**y**)) by lemma 2(i), whence $\neg$*S***x** by theorem 1(i). Since *M***x** by the definition of *M*, it follows that *C***x** by the definitions of *S* and *C*.

THEOREM 44. *Classes and intersection*
(i) *Let C*(**x**$\cap$**y**), *then C***x** $\wedge$ *C***y**.
(ii) *Let C*($\cap$**x***A*(**x**)), *then* $\forall$**z**(*A*(**z**)$\to$*C***z**).

PROOF OF (i). By the hypothesis, lemma 3(i) and the definitions of *C* and *S*, it follows that E!(**x**$\cap$**y**) and $\neg$*E*(**x**$\cap$**y**) and $\neg$*S*(**x**$\cap$**y**), whence *m***z**(**z**$\in$**x** $\wedge$ **z**$\in$**y**) by theorems 2(i) and 2(ii). Hence *M***x** $\wedge$ *M***y** by the definition of *M*, whence *C***x** $\wedge$ *C***y** by theorem 2(iv) and the definitions of *C* and *S*.

PROOF OF (ii). By the hypothesis, lemma 3(i) and the definitions of *C* and *S*, it follows that E!$\cap$**x***A*(**x**) and $\neg$*E*($\cap$**x***A*(**x**)) and $\neg$*S*($\cap$**x***A*(**x**)), whence *m***x**$\forall$**y**(*A*(**y**)$\to$**x**$\in$**y**) by theorems 19(i) and 19(ii). Hence $\forall$**y**(*A*(**y**)$\to$*M***y**) by the definition of *M*, whence $\forall$**z**(*A*(**z**)$\to$*C***z**) by theorem 19(iv) and the definitions of *C* and *S*.

THEOREM 45. *Classes and union*
(i) (*M***x** $\wedge$ *M***y** $\wedge$ (*C***x** $\vee$ *C***y**))$\leftrightarrow$*C*(**x**$\cup$**y**)
(ii) ($\forall$**y**(*A*(**y**)$\to$*M***y**) $\wedge$ ($\exists$**z**(*A*(**z**) $\wedge$ *C***z**) $\vee$ *C*(**z**:*A*(**z**))))$\leftrightarrow$*C*($\cup$**x***A*(**x**))

PROOF OF (i).
1 For the $\to$ half, suppose *M***x** $\wedge$ *M***y** $\wedge$ (*C***x** $\vee$ *C***y**). Then *M*(**x**$\cup$**y**) by theorem 20(i) and $\neg$*S***x** $\vee$ $\neg$*S***y** by the definitions of *C* and *S*, whence *C*(**x**$\cup$**y**) by the definitions of *C* and *S*, and theorem 20(ii).
2 For the $\leftarrow$ half, suppose *C*(**x**$\cup$**y**). Then *M*(**x**$\cup$**y**) and $\neg$*S*(**x**$\cup$**y**) by the definitions of *C* and *S*, whence *M***x** $\wedge$ *M***y** by theorem 20(i). Hence *C***x** $\vee$ *C***y** by the definitions of *C* and *S*, and theorem 20(ii).

PROOF OF (ii).

1 For the $\to$ half, suppose $\forall\mathbf{y}(A(\mathbf{y})\to M\mathbf{y}) \wedge (\exists\mathbf{z}(A(\mathbf{z}) \wedge C\mathbf{z}) \vee C(\mathbf{z}{:}A(\mathbf{z})))$. Then $(\forall\mathbf{y}(A(\mathbf{y})\to M\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z})) \vee (\forall\mathbf{y}(A(\mathbf{y})\to M\mathbf{y}) \wedge C(\mathbf{z}{:}A(\mathbf{z})))$, whence $(\forall\mathbf{y}(A(\mathbf{y})\to M\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z}))$ by lemmas 2(iv), 3(i) and 3(ii). Hence $M(\cup\mathbf{x}A(\mathbf{x}))$ by theorem 21(i). Suppose $\exists\mathbf{z}(A(\mathbf{z}) \wedge C\mathbf{z})$. Then $\neg\forall\mathbf{y}(A(\mathbf{y})\to S\mathbf{y})$ by the definitions of C and S, whence $\neg S(\cup\mathbf{x}A(\mathbf{x}))$ by theorem 21(ii). Suppose instead $C(\mathbf{z}{:}A(\mathbf{z}))$. Then $\neg S(\mathbf{z}{:}A(\mathbf{z}))$ by the definitions of C and S, whence $\neg S(\cup\mathbf{x}A(\mathbf{x}))$ by theorem 21(ii). So either way $\neg S(\cup\mathbf{x}A(\mathbf{x}))$, whence $C(\cup\mathbf{x}A(\mathbf{x}))$ by the definitions of C and S.

2 For the $\leftarrow$ half, suppose $C(\cup\mathbf{x}A(\mathbf{x}))$. Then $M(\cup\mathbf{x}A(\mathbf{x}))$ and $\neg S(\cup\mathbf{x}A(\mathbf{x}))$ by the definitions of C and S, whence $\forall\mathbf{y}(A(\mathbf{y})\to M\mathbf{y}) \wedge \exists\mathbf{z}A(\mathbf{z})$ by theorem 21(i), and $\neg\forall\mathbf{y}(A(\mathbf{y})\to S\mathbf{y}) \vee \neg S\mathbf{z}{:}A(\mathbf{z})$ by theorem 21(ii).

2 Suppose $\neg\forall\mathbf{y}(A(\mathbf{y})\to S\mathbf{y})$. Then $\exists\mathbf{z}(A(\mathbf{z}) \wedge \neg S\mathbf{z})$, whence $\exists\mathbf{z}(A(\mathbf{z}) \wedge C\mathbf{z})$ by the definitions of C and S; a fortiori $\exists\mathbf{z}(A(\mathbf{z}) \wedge C\mathbf{z}) \vee C\mathbf{z}{:}A(\mathbf{z})$.

3 Suppose instead $\neg S(\mathbf{z}{:}A(\mathbf{z}))$. If $\neg\mathsf{E}!(\mathbf{z}{:}A(\mathbf{z}))$ then $\forall\mathbf{z}(A(\mathbf{z})\to\neg E\mathbf{z})$ by lemma 2(iv), whence $\forall\mathbf{z}(A(\mathbf{z})\to C\mathbf{z})$ by the definition of C. Hence $\exists\mathbf{z}(A(\mathbf{z}) \wedge C\mathbf{z})$; a fortiori $\exists\mathbf{z}(A(\mathbf{z}) \wedge C\mathbf{z}) \vee C(\mathbf{z}{:}A(\mathbf{z}))$. If $\mathsf{E}!(\mathbf{z}{:}A(\mathbf{z}))$ then $\mathbf{z}{:}A(\mathbf{z})=\iota\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z})) \vee \mathbf{z}{:}A(\mathbf{z})=\mathbf{z}\cdot A(\mathbf{z})$ by lemmas 2(ii), 2(iii) and 2(iv). For a reductio suppose $\mathbf{z}{:}A(\mathbf{z})=\iota\mathbf{z}(E\mathbf{z} \wedge A(\mathbf{z}))$. Then $E(\mathbf{z}{:}A(\mathbf{z}))$, whence $U(\mathbf{z}{:}A(\mathbf{z})) \vee S(\mathbf{z}{:}A(\mathbf{z}))$ by lemma 3(iii). But also $A(\mathbf{z}{:}A(\mathbf{z}))$, whence $M(\mathbf{z}{:}A(\mathbf{z}))$. Hence $\neg U(\mathbf{z}{:}A(\mathbf{z}))$ by the definition of U, whence $S(\mathbf{z}{:}A(\mathbf{z}))$. Contradiction. Hence $\mathbf{z}{:}A(\mathbf{z})=\mathbf{z}\cdot A(\mathbf{z})$, whence $M(\mathbf{z}{:}A(\mathbf{z}))$ by lemma 4. Hence $C(\mathbf{z}{:}A(\mathbf{z}))$ by the definitions of C and S; a fortiori $\exists\mathbf{z}(A(\mathbf{z}) \wedge C\mathbf{z}) \vee C(\mathbf{z}{:}A(\mathbf{z}))$.

4 So either way $\exists\mathbf{z}(A(\mathbf{z}) \wedge C\mathbf{z}) \vee C(\mathbf{z}{:}A(\mathbf{z}))$. Since $\forall\mathbf{y}(A(\mathbf{y})\to M\mathbf{y})$ (from step 1), it follows that $\forall\mathbf{y}(A(\mathbf{y})\to M\mathbf{y}) \wedge (\exists\mathbf{z}(A(\mathbf{z}) \wedge C\mathbf{z}) \vee C(\mathbf{z}{:}A(\mathbf{z})))$

THEOREM 46. *Classes and pairing*

(i) *Let* $\mathsf{E}!\mathbf{x} \wedge \mathsf{E}!\mathbf{y} \wedge \mathbf{x}\neq\mathbf{y}$, *then* $\neg\mathsf{E}!|\mathbf{x}, \mathbf{y}| \leftrightarrow C\mathbf{x} \vee C\mathbf{y}$.

(ii) $\neg\exists\mathbf{x}\exists\mathbf{y}C|\mathbf{x}, \mathbf{y}|$

PROOF OF (i).

1 For the $\to$ half, suppose $\neg\mathsf{E}!|\mathbf{x}, \mathbf{y}|$. Then $\neg E\mathbf{x} \vee \neg E\mathbf{y}$ and $m\mathbf{x}U\mathbf{x}$ and $U\mathbf{x} \vee M\mathbf{x}$ and $U\mathbf{y} \vee M\mathbf{y}$ by the hypothesis, theorems 23(i) and 28(iv) and lemma 3(i). Suppose $\neg E\mathbf{x}$. Then $\neg U\mathbf{x} \wedge \neg S\mathbf{x}$ by lemma 3(iii), whence $M\mathbf{x}$. Hence $C\mathbf{x}$ by the definitions of C and S; a fortiori $C\mathbf{x} \vee C\mathbf{y}$. Suppose instead $\neg E\mathbf{y}$. Then $C\mathbf{x} \vee C\mathbf{y}$ follows by similar reasoning.

2 For the $\leftarrow$ half, suppose $C\mathbf{x} \vee C\mathbf{y}$. By theorem 23(i) and the definition of C, if $C\mathbf{x}$ then $\neg\mathsf{E}!|\mathbf{x}, \mathbf{y}|$, and if $C\mathbf{y}$ then $\neg\mathsf{E}!|\mathbf{x}, \mathbf{y}|$.

PROOF OF (ii). For a reductio suppose $C|\mathbf{x}, \mathbf{y}|$ for some $\mathbf{x}$, $\mathbf{y}$. Then $\mathsf{E}!|\mathbf{x}, \mathbf{y}|$ by lemmas 3(i) and 3(ii). Hence $S|\mathbf{x}, \mathbf{y}|$ by theorem 23(ii), whence $\neg C|\mathbf{x}, \mathbf{y}|$ by the definitions of C and S. Contradiction. Hence $\neg\exists\mathbf{x}\exists\mathbf{y}C|\mathbf{x}, \mathbf{y}|$.

THEOREM 47. *Classes and adjunction* $C\mathbf{x} \wedge E\mathbf{y} \leftrightarrow C(\mathbf{x}\oplus\mathbf{y})$

PROOF. By the definitions of C and S it follows that $C\mathbf{x} \wedge E\mathbf{y} \leftrightarrow M\mathbf{x} \wedge \neg S\mathbf{x} \wedge E\mathbf{y}$, whence $C\mathbf{x} \wedge E\mathbf{y} \leftrightarrow M(\mathbf{x}\oplus\mathbf{y}) \wedge \neg S(\mathbf{x}\oplus\mathbf{y})$ by theorems 24(i) and 24(ii). Hence $C\mathbf{x} \wedge E\mathbf{y} \leftrightarrow C(\mathbf{x}\oplus\mathbf{y})$ by the definitions of C and S.

THEOREM 48. *Classes, power and power-plus*
(i) $C\mathbf{x}\leftrightarrow C(P(\mathbf{x}))$
(ii) $C\mathbf{x}\leftrightarrow C(P^{+}(\mathbf{x}))$

PROOF OF (i). By the definitions of C and S it follows that $C\mathbf{x} \leftrightarrow M\mathbf{x} \wedge \neg S\mathbf{x}$, whence $C\mathbf{x} \leftrightarrow M(P(\mathbf{x})) \wedge \neg S(P(\mathbf{x}))$ by theorems 25(iii) and 25(iv). Hence $C\mathbf{x}\leftrightarrow C(P(\mathbf{x}))$ by the definitions of C and S.

PROOF OF (ii). By the definitions of C and S it follows that $C\mathbf{x} \leftrightarrow M\mathbf{x} \wedge \neg S\mathbf{x}$, whence $C\mathbf{x} \leftrightarrow M(P^{+}(\mathbf{x})) \wedge \neg S(P^{+}(\mathbf{x}))$ by theorems 26(i) and 26(ii). Hence $C\mathbf{x}\leftrightarrow C(P^{+}(\mathbf{x}))$ by the definitions of C and S.

THEOREM 49. *Classes and reproductivity*
(i) $C\mathbf{x}\leftrightarrow\exists\mathbf{y}(C\mathbf{y} \wedge \mathbf{y}\subset\mathbf{x})$
(ii) $C\mathbf{x}\leftrightarrow(M\mathbf{x} \wedge \forall\mathbf{y}((S\mathbf{y} \wedge \mathbf{y}\subseteq\mathbf{x})\rightarrow\exists\mathbf{z}(S\mathbf{z} \wedge \mathbf{y}\subset\mathbf{z} \wedge \mathbf{z}\subset\mathbf{x})))$

PROOF OF (i).

1 For the $\rightarrow$ half, suppose $C\mathbf{x}$. Then $M\mathbf{x} \wedge \neg S\mathbf{x} \wedge m\mathbf{z}\ \mathbf{z}\in\mathbf{x}$ by the definitions of C, S and M. For a reductio suppose $\mathbf{z}_1\neq\mathbf{z}_2 \wedge \mathbf{z}_1\in\mathbf{x} \wedge \mathbf{z}_2\in\mathbf{x} \wedge \forall\mathbf{z}_3(\mathbf{z}_3\in\mathbf{x}\rightarrow\mathbf{z}_3=\mathbf{z}_1 \vee \mathbf{z}_3=\mathbf{z}_2)$, for some $\mathbf{z}_1$, $\mathbf{z}_2$. Then $\mathbf{z}\in\mathbf{x}\leftrightarrow(\mathbf{z}=\mathbf{z}_1 \vee \mathbf{z}=\mathbf{z}_2)$. Also by lemma 5(i), $E\mathbf{z}_1 \wedge E\mathbf{z}_2$. Hence by theorems 23(i) and 23(ii), and the definition of S, it follows that $\mathsf{E}!|\mathbf{z}_1, \mathbf{z}_2| \wedge S|\mathbf{z}_1, \mathbf{z}_2| \wedge M|\mathbf{z}_1, \mathbf{z}_2|$, whence $\mathbf{z}\in|\mathbf{z}_1, \mathbf{z}_2|\leftrightarrow(\mathbf{z}=\mathbf{z}_1 \vee \mathbf{z}=\mathbf{z}_2)$ by lemma 4 and the definition of $|\mathbf{z}_1, \mathbf{z}_2|$. Hence $\mathbf{x}=|\mathbf{z}_1, \mathbf{z}_2|$ by axiom 1(ii), whence $S\mathbf{x}$. Contradiction. Hence $\mathbf{z}_1\neq\mathbf{z}_2 \wedge \mathbf{z}_1\neq\mathbf{z}_3 \wedge \mathbf{z}_2\neq\mathbf{z}_3 \wedge \mathbf{z}_1\in\mathbf{x} \wedge \mathbf{z}_2\in\mathbf{x} \wedge \mathbf{z}_3\in\mathbf{x}$, for some $\mathbf{z}_1$, $\mathbf{z}_2$, $\mathbf{z}_3$. Hence $m\mathbf{z}(E\mathbf{z} \wedge \mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\neq\mathbf{z}_3)$ by lemma 5(i), whence $\mathsf{E}!(\mathbf{z}\cdot\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\neq\mathbf{z}_3)$ by lemma 2(i), and $M(\mathbf{z}\cdot\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\neq\mathbf{z}_3) \wedge \forall\mathbf{y}(\mathbf{y}\in(\mathbf{z}\cdot\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\neq\mathbf{z}_3)\leftrightarrow(\mathbf{y}\in\mathbf{x} \wedge \mathbf{y}\neq\mathbf{z}_3))$ by lemmas 4 and 5(i). Hence $(\mathbf{z}\cdot\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\neq\mathbf{z}_3)\subseteq\mathbf{x}$ by the definition of $\subseteq$. Since $\mathbf{z}_3\notin(\mathbf{z}\cdot\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\neq\mathbf{z}_3)$ and $\mathbf{z}_3\in\mathbf{x}$, it follows that $(\mathbf{z}\cdot\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\neq\mathbf{z}_3)\subset\mathbf{x}$ by the definition of $\subset$.

2 For a reductio suppose $S(\mathbf{z}\cdot\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\neq\mathbf{z}_3)$. Then $S((\mathbf{z}\cdot\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\neq\mathbf{z}_3)\oplus\mathbf{z}_3) \wedge M((\mathbf{z}\cdot\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\neq\mathbf{z}_3)\oplus\mathbf{z}_3) \wedge \mathsf{E}!((\mathbf{z}\cdot\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\neq\mathbf{z}_3)\oplus\mathbf{z}_3)$ by theorem 24(ii), lemmas 3(i) and 5(i), and the definition of S. Hence $\forall\mathbf{y}(\mathbf{y}\in((\mathbf{z}\cdot\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\neq\mathbf{z}_3)\oplus\mathbf{z}_3)\leftrightarrow\mathbf{y}\in\mathbf{x})$ by lemmas 4 and 5(i) and the definition of $\oplus$, whence $((\mathbf{z}\cdot\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\neq\mathbf{z}_3)\oplus\mathbf{z}_3)=\mathbf{x}$ by axiom 1(ii). Hence $S\mathbf{x}$. Contradiction. Hence $C(\mathbf{z}\cdot\mathbf{z}\in\mathbf{x} \wedge \mathbf{z}\neq\mathbf{z}_3)$ by the definition of C and S, whence $\exists\mathbf{y}(\mathbf{y}\subset\mathbf{x} \wedge C\mathbf{y})$.

3 For the $\leftarrow$ half, suppose $C\mathbf{y} \wedge \mathbf{y}\subset\mathbf{x}$ for some $\mathbf{y}$. Then $C\mathbf{x}$ by the definition of $\subset$ and theorem 43(i).

PROOF OF (ii).

1 For the $\rightarrow$ half, suppose $C\mathbf{x}$. Then $M\mathbf{x} \wedge \neg S\mathbf{x} \wedge \mathsf{E}!\mathbf{x}$ by the definitions of C and S, and lemma 3(i). For a reductio suppose for some $\mathbf{y_1}$, $S\mathbf{y_1} \wedge \mathbf{y_1}\subseteq\mathbf{x} \wedge \neg\exists\mathbf{z}(S\mathbf{z} \wedge \mathbf{y_1}\subset\mathbf{z} \wedge \mathbf{z}\subset\mathbf{x})$. Then $M\mathbf{y_1}$ by the definition of S. Since $\neg S\mathbf{x} \wedge S\mathbf{y_1}$, it follows that $\mathbf{y_1}\neq\mathbf{x}$, whence $\mathbf{y_1}\subset\mathbf{x}$ by the definition of $\subset$. Hence for some $\mathbf{z_1}$, $\mathbf{z_1}\in\mathbf{x} \wedge \mathbf{z_1}\notin\mathbf{y_1} \wedge E\mathbf{z_1}$ by lemmas 5(i) and 6(iii), whence $S(\mathbf{y_1}\oplus\mathbf{z_1}) \wedge M(\mathbf{y_1}\oplus\mathbf{z_1}) \wedge \mathsf{E}!(\mathbf{y_1}\oplus\mathbf{z_1})$ by theorem 24(ii), lemma 3(i), and the definition of S. Hence $(\mathbf{z_2}\in\mathbf{y_1} \vee \mathbf{z_2}=\mathbf{z_1})\leftrightarrow\mathbf{z_2}\in\mathbf{y_1}\oplus\mathbf{z_1}$ by lemmas 4 and 5(i), and the definition of $\oplus$, whence $\mathbf{y_1}\subseteq\mathbf{y_1}\oplus\mathbf{z_1}$ by the definition of $\subseteq$. Since $\mathbf{z_1}\notin\mathbf{y_1} \wedge \mathbf{z_1}\in\mathbf{y_1}\oplus\mathbf{z_1}$, it follows that $\mathbf{y_1}\neq\mathbf{y_1}\oplus\mathbf{z_1}$, whence $\mathbf{y_1}\subset\mathbf{y_1}\oplus\mathbf{z_1}$ by the definition of $\subset$.

2 Since $\mathbf{y_1}\subseteq\mathbf{x} \wedge \mathbf{z_1}\in\mathbf{x}$, it follows that $\mathbf{z_2}\in\mathbf{y_1}\oplus\mathbf{z_1}\rightarrow\mathbf{z_2}\in\mathbf{x}$ by the definition of $\subseteq$, whence $\mathbf{y_1}\oplus\mathbf{z_1}\subseteq\mathbf{x}$ by the definition of $\subseteq$. Since $S(\mathbf{y_1}\oplus\mathbf{z_1}) \wedge \neg S\mathbf{x}$, it follows that $\mathbf{y_1}\oplus\mathbf{z_1}\neq\mathbf{x}$, whence $\mathbf{y_1}\oplus\mathbf{z_1}\subset\mathbf{x}$ by the definition of $\subset$. Hence $\exists\mathbf{z}(S\mathbf{z} \wedge \mathbf{y_1}\subset\mathbf{z} \wedge \mathbf{z}\subset\mathbf{x})$. Contradiction. Hence $\forall\mathbf{y}((S\mathbf{y} \wedge \mathbf{y}\subseteq\mathbf{x})\rightarrow\exists\mathbf{z}(S\mathbf{z} \wedge \mathbf{y}\subset\mathbf{z} \wedge \mathbf{z}\subset\mathbf{x}))$.

3 For the $\leftarrow$ half, suppose $M\mathbf{x} \wedge \forall\mathbf{y}((S\mathbf{y} \wedge \mathbf{y}\subseteq\mathbf{x})\rightarrow\exists\mathbf{z}(S\mathbf{z} \wedge \mathbf{y}\subset\mathbf{z} \wedge \mathbf{z}\subset\mathbf{x}))$. For a reductio suppose $\neg C\mathbf{x}$. Then $S\mathbf{x} \wedge \mathbf{x}\subseteq\mathbf{x}$ by the definitions of C and S, and lemma 6(ii), whence for some $\mathbf{z}$, $S\mathbf{z} \wedge \mathbf{x}\subset\mathbf{z} \wedge \mathbf{z}\subset\mathbf{x}$. Hence for some $\mathbf{z_1}$, $\mathbf{z_1}\in\mathbf{z} \wedge \mathbf{z_1}\notin\mathbf{x}$ by lemma 6(iii). But since $\mathbf{z_1}\in\mathbf{z} \wedge \mathbf{z}\subset\mathbf{x}$, it follows that $\mathbf{z_1}\in\mathbf{x}$ by the definitions of $\subset$ and $\subseteq$. Contradiction. Hence $C\mathbf{x}$.

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